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WEAK ∞ -CATEGORIES VIA TERMINAL COALGEBRAS

EUGENIA CHENG AND TOM LEINSTER

ABSTRACT.

Higher categorical structures are often defined by induction on dimension, which *a priori* produces only finite-dimensional structures. In this paper we show how to extend such definitions to infinite dimensions using the theory of terminal coalgebras, and we apply this method to Trimble’s notion of weak n -category. Trimble’s definition makes explicit the relationship between n -categories and topological spaces; our extended theory produces a definition of Trimble ∞ -category and a fundamental ∞ -groupoid construction.

Furthermore, terminal coalgebras are often constructed as limits of a certain type. We prove that the theory of Batanin–Leinster weak ∞ -categories arises as just such a limit, justifying our approach to Trimble ∞ -categories. In fact we work at the level of monads for ∞ -categories, rather than ∞ -categories themselves; this requires more sophisticated technology but also provides a more complete theory of the structures in question.

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Introduction

The main aim of this work is to give a coinductive definition of Trimble-style ∞ -categories. We do this using the theory of terminal coalgebras for endofunctors.

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In 1999, Trimble proposed a definition of weak n -category which he called “flabby” n -category [14]. The definition proceeds by iterated enrichment. It is well known that *strict* n -categories can be defined by iterated enrichment, that is, for $n \geq 1$, a strict n -category is precisely a category enriched in strict $(n - 1)$ -categories. However, for *weak* n -categories, a notion of weak enrichment is required.

Trimble does not attempt a fully general definition of weak enrichment, but focusses on the motivation from topology: the idea that every topological space X should give rise to a weak ∞ -category, its so-called fundamental ∞ -groupoid $\Pi_\infty X$. This was Trimble’s main goal, although he only dealt with the finite-dimensional case, and to this end, he built the fundamental n -groupoid construction into the definition of higher-dimensional category.

Given a space X , it is clear what the underlying globular set of its fundamental ∞ -groupoid $\Pi_\infty X$ should be: the 0-cells are the points of X , the 1-cells are the paths, the 2-cells are the homotopies of paths, and so on. This underlying data comes equipped with various operations, arising, for instance, from composition of paths and homotopies; the question of defining fundamental ∞ -groupoids comes down to how to express these operations.

In Trimble’s definition, the operations available in an abstract higher-dimensional category are dictated by the homotopical operations generically available in topological spaces as above. Thus, the definition depends inherently on the structure of **Top**, a suitable category of topological spaces.

The definition uses operads to parametrise the weakly associative composition in the higher-dimensional structures. It is well-known how operads can be used to describe homotopy monoids, in which the multiplication is associative and unital only up to homotopy. Trimble uses a generalised version of this to make enriched categories in which composition is associative and unital only “up to homotopy”. The idea is as follows.

Given a symmetric monoidal category \mathcal{V} and an operad P in \mathcal{V} , we may consider both \mathcal{V} -categories and P -algebras. These concepts can be amalgamated to obtain a third concept: that of (\mathcal{V}, P) -category, or category “enriched in \mathcal{V} and weakened by P ”. This concept has been defined and used under several names by several authors [18, 4]. An explanation with examples can be found in [7]; an alternative point of view is given in [4], in which it is made particularly clear the sense in which this is a form of weak enrichment.

In order to iterate enrichment of this form, it seems that we need a *series* of operads, one for each stage of the enrichment. While this is true *a priori*, Trimble cleverly produces these operads as part of the enrichment. For each n he defines not just a category $n\text{-Cat}$ of n -categories, but a fundamental n -groupoid functor

$$\Pi_n: \mathbf{Top} \longrightarrow n\text{-Cat}.$$

He then only needs to specify one operad E in **Top**; this produces a series of operads $\Pi_n E$ in $n\text{-Cat}$ which can be used for the iterated enrichment.

As this definition is inductive, Trimble only defined a notion of n -category for finite n . However by taking an appropriate limit we can define a notion of ∞ -category for this theory.

We emphasise that we are using the term “ ∞ -category” in its fully general sense (often also called “ ω -categories”). We do not require that any of the cells in an ∞ -category are in any way invertible. This is in contrast to the restricted meaning given to the term by some authors, who use it as a synonym for what are properly called $(\infty, 1)$ -categories (∞ -categories in which all cells of dimension greater than 1 are weakly invertible). Thus, 2-categories, 3-categories and so on are all special cases of our ∞ -categories.

Our limit-based approach to defining Trimble-style weak ∞ -categories is motivated by analogy with

1. strict ∞ -categories, and
2. Batanin/Leinster weak ∞ -categories.

In both cases we can build ∞ -categories as a limit of n -dimensional truncations; however in the weak case some delicacy is required as the truncation of a weak ∞ -category is not a weak n -category because of coherence issues at the top dimension.

We can take further inspiration from strict ∞ -categories as the construction can also be made via an enrichment endofunctor

$$\mathcal{V} \longmapsto \mathcal{V}\text{-}\mathbf{Cat}.$$

The limit in question is then the terminal coalgebra for this endofunctor, via Adámek’s theorem ([1], see [2] for a modern survey); the limit can be thought of as an “infinite iteration” of the endofunctor. The theorem tells us that under certain conditions, the terminal coalgebra can be computed as the limit of the following sequential diagram

$$\dots \xrightarrow{F^3!} F^3 1 \xrightarrow{F^2!} F^2 1 \xrightarrow{F!} F 1 \xrightarrow{!} 1 \quad (1)$$

The point is that in our examples, this diagram consists of (up to isomorphism) the n -dimensional truncations of our infinite-dimensional structure, so Adámek’s Theorem gives us the terminal coalgebra as a limit of certain n -dimensional structures, which is what we would like the infinite-dimensional structure to be.

The Batanin-Leinster definition is not by iterated enrichment (at least, not directly) but the Trimble definition is, in the appropriate weak sense. So our aim is:

1. Define the category of Trimble ∞ -categories as a limit of n -dimensional truncations.
2. Express this limit as the terminal coalgebra for a “weak enrichment” endofunctor.

There are three further subtleties. First, we prefer to work at the level of theories, so instead of simply constructing the *category* of Trimble ∞ -categories from the categories of n -dimensional truncations, we wish to construct the *monad* for Trimble ∞ -categories from the n -dimensional monads.

Secondly, Trimble’s construction of n -categories is dependent on, and given simultaneously with, the “fundamental n -groupoid” functor

$$\Pi_n: \mathbf{Top} \longrightarrow n\text{-}\mathbf{Cat}$$

at each stage of the enrichment. This means that we need to work in a base category that expresses this data, that is, some sort of slice category. Roughly speaking, instead of having an endofunctor for enrichment

$$\mathcal{V} \mapsto \mathcal{V}\text{-Cat}$$

we need one that includes Π , as in

$$\mathbf{Top} \xrightarrow{\Pi} \mathcal{V} \quad \mapsto \quad \mathbf{Top} \xrightarrow{\Pi^+} \mathcal{V}\text{-Cat}.$$

So this endofunctor seems to act on $\mathbf{Top}/\mathbf{CAT}$; for the monad version the situation needs to be adjusted accordingly.

Finally, in order for the inductive constructions to go through, we need some conditions on our categories and functors. To construct the sort of enriched categories we require, we need to enrich in categories with finite products. To construct the *monads* that freely generate such enriched categories, we further need small coproducts and some distributivity conditions. Thus we need to restrict the categories \mathbf{CAT} and \mathbf{MND} (a category of monads) variously. This creates some technical questions when we come to take limits, but it is not too hard to check that this does not actually create any obstacles. We include these technical results in an Appendix in order not to interrupt the flow of the paper; at a first approximation the main body of the paper can be read under the understanding that all limits required can be computed as expected.

The paper has two analogous halves, for the strict and weak cases. As a preliminary, in Section 1 we recall the basic theory of terminal coalgebras that we will be using throughout. We then apply this theory to analyse strict ∞ -categories in three steps. In Section 2 we study the underlying data, ∞ -graphs, in Section 3 the categories of n -categories, and in Section 4 the monads for n -categories. This part serves partly as a warm-up for the more technically involved constructions of the second half, and partly as a basis—the second half will be not only analogous but will also build technically on these basic results.

In Section 5 we take a brief interlude from terminal coalgebras and study Batanin-Leinster weak ∞ -categories by means of limits of n -dimensional truncations; this section thus assumes the most background. Then we use terminal coalgebras to study Trimble ∞ -categories analogously to Sections 2–4: in Section 6 we study the underlying data, ∞ -graphs, in Section 7 the categories of n -categories, and in Section 8 the monads for n -categories.

In each case the analysis follows broadly the same steps (though of course as things get more complicated, more definitions and intermediate lemmas are required):

1. Define the category in which we work.
2. Define the endofunctor we study.
3. Define the n -dimensional structures we use, and truncation maps from n to $n - 1$ dimensions.

4. Show that these can be obtained by applying our endofunctor to the terminal object repeatedly, giving the diagram that appears in Adámek's Theorem (1).
5. Use Adámek's Theorem to obtain infinite-dimensional structure as a limit of n -dimensional structures, hence as a terminal coalgebra for the endofunctor in question.

Some of the work in this paper building up to the coalgebra results overlaps with the work of Weber [22] which was developed independently at around the same time. Weber studies the enriched graph endofunctor and gives an inductive definition of the strict n -category monads, and the Trimble n -category monads, and he also makes widespread use of "The formal theory of monads" [20]. However the emphasis of [22] and the related series of papers is different: there the aim is to give a framework for general processes of enrichment analogous to the construction and use of the Gray tensor product. The aim of the current work is specifically to produce infinite-dimensional structures from finite-dimensional ones. It is quite likely that Weber's work can be extended to give a concise approach to infinite-dimensional constructions in the manner of the present paper, but this is beyond our current scope.

ASSUMED BACKGROUND.

NOTATION AND TERMINOLOGY. We will use various subscripts throughout on both our categories **CAT** and **MND**, and our endofunctors F and D . Lower case subscript "p" indicates finite products, and "d" indicates finite products and small coproducts with distributivity. On our endofunctors we will use "G", "C" and "M" to indicate when we are studying underlying graphs, n -categories or monads respectively. Our six main examples are summed up in the following table; although some of the notation will not be explained until the main part of the paper we feel it will be useful to sum up the structures at this point.

	strict	Trimble
underlying data	Section 2 F_G on CAT $\mathcal{V} \mapsto \mathcal{V}\text{-}\mathbf{Gph}$	Section 6 D_G on Top/CAT $(\mathcal{V}, \Pi) \mapsto (\mathcal{V}\text{-}\mathbf{Gph}, \Pi^+)$
categories of ∞ -categories	Section 3 F_C on CAT_p $\mathcal{V} \mapsto \mathcal{V}\text{-}\mathbf{Cat}$	Section 7 D_C on Top/CAT_p $(\mathcal{V}, \Pi) \mapsto ((\mathcal{V}, \Pi E)\text{-}\mathbf{Cat}, \Pi^+)$
monads for ∞ -categories	Section 4 F_M on MND_d $(\mathcal{V}, T) \mapsto (\mathcal{V}\text{-}\mathbf{Gph}, T^+)$	Section 8 D_M on (Top, 1)/MND_d $(\mathcal{V}, T, \Pi) \mapsto (\mathcal{V}\text{-}\mathbf{Gph}, T^+, \Pi^+)$

We will often need to refer to structures defined by equations that hold only up to coherent isomorphism or equivalence. In the literature, a complex system of modifiers has evolved, all expressing this same condition, including “weak”, “strong”, “pseudo”, and no modifier at all. We consistently use “weak” throughout, which has the disadvantage of differing from some of the literature, but the advantages of simplicity and uniformity.

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1. Background on terminal coalgebras

1.1. DEFINITION. A **coalgebra** for an endofunctor $F: \mathcal{C} \longrightarrow \mathcal{C}$ consists of

- an object $A \in \mathcal{C}$

- a morphism
$$\begin{array}{c} A \\ \downarrow \\ FA \end{array}$$

satisfying no axioms. Coalgebras for F form a category with the obvious morphisms

$$\begin{array}{ccc} A & \xrightarrow{h} & B \\ \downarrow & & \downarrow \\ FA & \xrightarrow{Fh} & FB \end{array}$$

so we can look for terminal coalgebras.

1.2. **EXAMPLE.** Given a set M we have an endofunctor

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{M \times -} & \mathbf{Set} \\ A & \longmapsto & M \times A. \end{array}$$

A coalgebra for this is a function

$$\begin{array}{ccc} A & \xrightarrow{(m,f)} & M \times A \\ a & \longmapsto & (m(a), f(a)). \end{array}$$

The terminal coalgebra is given by the set $M^{\mathbb{N}}$ of “infinite words”

$$(m_0, m_1, m_2, \dots)$$

in M . The structure map

$$\begin{array}{c} M^{\mathbb{N}} \\ \downarrow \\ M \times M^{\mathbb{N}} \end{array}$$

is given by a canonical isomorphism. To show that this is terminal, given any coalgebra

$$\begin{array}{c} A \\ \downarrow (m,f) \\ M \times A \end{array}$$

we need a unique map

$$\begin{array}{ccc} A & \xrightarrow{t} & M^{\mathbb{N}} \\ \downarrow & & \downarrow \\ M \times A & \xrightarrow{1 \times t} & M \times M^{\mathbb{N}} \end{array}$$

and we have

$$t: a \longmapsto (m(a), m(f(a)), m(f^2(a)), m(f^3(a)), \dots).$$

1.3. **EXAMPLE.** Let F be the free monoid monad on \mathbf{Set} . The terminal coalgebra is given by the set Tr^{∞} of “infinite trees of finite arity”. The structure map

$$\begin{array}{c} \mathrm{Tr}^{\infty} \\ \downarrow \\ F(\mathrm{Tr}^{\infty}) = \text{“finite strings of infinite trees”} \end{array}$$

is again given by a canonical isomorphism. For details, see, for instance, 3.9 and 3.10 of [2].

1.4. LEMMA. [Lambek [13]] *Let F be an endofunctor of a category. If*

$$\begin{array}{c} A \\ \downarrow f \\ FA \end{array}$$

is a terminal coalgebra for F then f is an isomorphism.

1.5. THEOREM. [Adámek [1]] *Let F be an endofunctor on a category with a terminal object 1 . Suppose that the limit of the following diagram exists and is preserved by F :*

$$\dots \xrightarrow{F^3!} F^3 1 \xrightarrow{F^2!} F^2 1 \xrightarrow{F!} F 1 \xrightarrow{!} 1$$

Then this limit is (the underlying object of) a terminal coalgebra for F .

We now revisit the previous two examples using this theorem.

1.6. EXAMPLE. (cf. Example 1.2) Given a set M we considered the endofunctor

$$\begin{array}{ccc} \mathbf{Set} & \xrightarrow{M \times -} & \mathbf{Set} \\ A & \mapsto & M \times A \end{array}$$

and saw that the terminal coalgebra was given by the set $M^{\mathbb{N}}$ of “infinite words” in M . Using Theorem 1.5 we can construct a terminal coalgebra as the limit of

$$\dots \xrightarrow{F^3!} F^3 1 \xrightarrow{F^2!} F^2 1 \xrightarrow{F!} F 1 \xrightarrow{!} 1$$

For the functor F in this example, this diagram becomes

$$\dots \xrightarrow{M^3 \times !} M^3 \times 1 \xrightarrow{M^2 \times !} M^2 \times 1 \xrightarrow{M \times !} M \times 1 \xrightarrow{!} 1$$

which is isomorphic to

$$\dots \xrightarrow{M^3} M^3 \xrightarrow{M^2} M^2 \xrightarrow{M} M \xrightarrow{!} 1$$

and taking the limit of this diagram does indeed give infinite words in M .

Example 1.3 works similarly.

1.7. EXAMPLE. Let **CAT** be the category of locally small categories and functors between them. For any locally small category \mathcal{V} , let $\mathcal{V}\text{-Gph}$ be the (locally small) category of \mathcal{V} -graphs and their morphisms. A \mathcal{V} -graph A consists of

- a set **ob** A
- for all $x, y \in \mathbf{ob}A$ an object $A(x, y) \in \mathcal{V}$.

A morphism $f: A \rightarrow B$ of \mathcal{V} -graphs is given by a function $f: \mathbf{ob} A \rightarrow \mathbf{ob} B$ and for all pairs of objects $a, a' \in \mathbf{ob} A$, a morphism $A(a, a') \rightarrow B(fa, fa')$ in \mathcal{V} .

There is an endofunctor

$$\begin{aligned} (\)\text{-}\mathbf{Gph} &: \mathbf{CAT} \longrightarrow \mathbf{CAT} \\ \mathcal{V} &\longmapsto \mathcal{V}\text{-}\mathbf{Gph} \\ H &\longmapsto H_* \end{aligned}$$

where H_* “acts locally as H ”. That is, given a functor $H: \mathcal{V} \rightarrow \mathcal{W}$ and a \mathcal{V} -graph A , the \mathcal{W} -graph H_*A has the same objects as A , and

$$(H_*A)(a, a') = H(A(a, a')).$$

This functor has a terminal coalgebra given by the category $\infty\text{-}\mathbf{Gph}$ of ∞ -graphs, which is equivalent to the category \mathbf{GSet} of globular sets. We will study this example in detail in the next section.

1.8. EXAMPLE. We learnt the following example from Carlos Simpson. Write \mathbf{CAT}_p for the category of locally small finite product categories and finite product preserving functors. There is an endofunctor

$$\begin{aligned} \mathbf{CAT}_p &\longrightarrow \mathbf{CAT}_p \\ \mathcal{V} &\longmapsto \mathcal{V}\text{-}\mathbf{Cat} \\ H &\longmapsto H_*. \end{aligned}$$

This has a terminal coalgebra given by the category $\infty\text{-}\mathbf{Cat}$ of *strict* ∞ -categories. Section 3 consists of a detailed study of this example, as both motivation and warm-up for the main part of the paper.

The general idea, then, is that terminal coalgebras give us a way of constructing infinite versions of gadgets whose finite versions we can construct simply by induction. Our aim is to apply this to Trimble’s version of weak n -categories.

2. ∞ -graphs

This section contains the simplest of our terminal coalgebra theorems. It acts as a warm-up, in that the arguments in the later sections are similar to the arguments here but take place in more sophisticated settings. Some later sections will also depend on the theorem proved here.

Our purpose is to characterise $\infty\text{-}\mathbf{Gph}$ as a terminal coalgebra. As ∞ -graphs give the underlying data for all our ∞ -categories (strict and weak), this construction will be the basis of all further constructions. We use the following endofunctor, but introduce notation to fit with the overall system of the paper.

2.1. DEFINITION. We write F_G for the endofunctor

$$(\)\text{-}\mathbf{Gph}: \mathbf{CAT} \longrightarrow \mathbf{CAT}$$

defined in Example 1.7.

In fact, F_G is an endo-2-functor on the 2-category \mathbf{CAT} , with the action of natural transformations defined in the obvious way. For the most part, though, we will continue to treat \mathbf{CAT} as a 1-category.

A \mathcal{V} -graph is the underlying data for a \mathcal{V} -category, and an n -graph is the underlying data for an n -category. We define n -graphs by iterated enrichment as follows.

2.2. DEFINITION. For all $n \geq 0$ we define the category $n\text{-}\mathbf{Gph}$ of n -graphs as follows:

- $0\text{-}\mathbf{Gph} = \mathbf{Set}$, and
- for all $n \geq 1$, $n\text{-}\mathbf{Gph} = F_G((n-1)\text{-}\mathbf{Gph}) = ((n-1)\text{-}\mathbf{Gph})\text{-}\mathbf{Gph}$.

We also define for all $n \geq 1$ the **truncation** functor

$$U_n: n\text{-}\mathbf{Gph} \longrightarrow (n-1)\text{-}\mathbf{Gph},$$

as follows.

- $U_1: 1\text{-}\mathbf{Gph} \longrightarrow \mathbf{Set}$ is the functor that sends a graph to its set of objects, and
- for all $n \geq 2$, $U_n = F_G(U_{n-1}) = U_{n-1*}$.

Alternatively, n -graphs can also be understood as n -globular sets; the categories of such are equivalent. This is a little tangential to our main argument so we will discuss it at the end of the section.

It is straightforward to make an infinite-dimensional version of n -globular sets directly; this is the category \mathbf{GSet} of globular sets. However for the infinite-dimensional version of $n\text{-}\mathbf{Gph}$ we cannot proceed by induction. Instead we define infinite-dimensional graphs as follows. It is the simplest of our various coinductive constructions.

2.3. DEFINITION. The category $\infty\text{-}\mathbf{Gph}$ of ∞ -graphs is the limit of the diagram

$$\cdots \xrightarrow{U_3} 2\text{-}\mathbf{Gph} \xrightarrow{U_2} 1\text{-}\mathbf{Gph} \xrightarrow{U_1} 0\text{-}\mathbf{Gph}$$

in \mathbf{CAT} . We write $\mathrm{tr}_n: \infty\text{-}\mathbf{Gph} \longrightarrow n\text{-}\mathbf{Gph}$ for the n th projection of the limit cone, and refer to tr_n (as well as U_n) as **truncation**.

Thus the limit cone looks like

$$\begin{array}{ccccccc} & & \infty\text{-}\mathbf{Gph} & & & & \\ & \swarrow & & \searrow & & \searrow & \\ & & & & & \searrow & \\ \cdots & \xrightarrow{U_2} & 2\text{-}\mathbf{Gph} & \xrightarrow{U_2} & 1\text{-}\mathbf{Gph} & \xrightarrow{U_1} & 0\text{-}\mathbf{Gph} \\ & & \swarrow & & \swarrow & & \\ & & \mathrm{tr}_2 & & \mathrm{tr}_1 & & \mathrm{tr}_0 \end{array}$$

and we may express an ∞ -graph as an infinite sequence

$$(\dots, X_2, X_1, X_0)$$

where each X_n is an n -graph and is the truncation of X_{n+1} .

2.4. THEOREM. *The category $\infty\text{-Gph}$ is the terminal coalgebra for the endofunctor*

$$F_G: \mathbf{CAT} \longrightarrow \mathbf{CAT}.$$

PROOF. We use Adámek's theorem. We need to consider the limit in \mathbf{CAT} of

$$\dots \xrightarrow{F_G^3!} F_G^3 \mathbb{1} \xrightarrow{F_G^2!} F_G^2 \mathbb{1} \xrightarrow{F_G!} F_G \mathbb{1} \quad (2)$$

where $\mathbb{1}$ is the terminal object of \mathbf{CAT} . We need to show

1. the limit is $\infty\text{-Gph}$, and
2. the limit is preserved by F_G .

First we note that the above diagram (2) is isomorphic to the one whose limit defines $\infty\text{-Gph}$; formally, there are canonical isomorphisms $(I_n)_{n \geq 0}$ such that the following diagram commutes:

$$\begin{array}{ccccccc} \dots & \xrightarrow{F_G^3!} & F_G^3 \mathbb{1} & \xrightarrow{F_G^2!} & F_G^2 \mathbb{1} & \xrightarrow{F_G!} & F_G \mathbb{1} \\ & & \downarrow \cong I_2 & & \downarrow \cong I_1 & & \downarrow \cong I_0 \\ \dots & \xrightarrow{U_3} & \mathbf{2-Gph} & \xrightarrow{U_2} & \mathbf{1-Gph} & \xrightarrow{U_1} & \mathbf{0-Gph} \end{array}$$

This is true by a straightforward induction: we have

$$F_G \mathbb{1} = \mathbb{1}\text{-Gph} \cong \mathbf{Set} = \mathbf{0-Gph},$$

giving the isomorphism I_0 , and we put $I_n = F_G^n(I_0)$. The rightmost square of the diagram commutes; hence by induction, the whole diagram commutes.

This shows that the limit we need to take is over the following diagram

$$\dots \xrightarrow{U_3} \mathbf{2-Gph} \xrightarrow{U_2} \mathbf{1-Gph} \xrightarrow{U_1} \mathbf{0-Gph}$$

and this is by definition $\infty\text{-Gph}$.

We now show that F_G preserves this limit. (In fact, F_G preserves all connected limits. This can be proved by a similar argument, and also follows from the fact that F_G is a local right adjoint, as explained in Section 3 of Weber [22].) We need to show that $(\infty\text{-Gph})\text{-Gph}$ is the limit of the following diagram

$$\dots \longrightarrow (\mathbf{2-Gph})\text{-Gph} \xrightarrow{U_{2*}} (\mathbf{1-Gph})\text{-Gph} \xrightarrow{U_{1*}} (\mathbf{0-Gph})\text{-Gph}.$$

Let X be an object of the limit, so X consists of an infinite sequence

$$(\dots, X_2, X_1, X_0)$$

with $X_n \in (n\text{-}\mathbf{Gph})\text{-}\mathbf{Gph}$ and

$$(U_{n+1})_* X_{n+1} = X_n \quad (3)$$

for all $n \geq 0$. On objects this gives

$$\mathbf{ob} X_{n+1} = \mathbf{ob}((U_{n+1})_* X_{n+1}) = \mathbf{ob} X_n$$

for all $n \geq 0$, so the graphs X_n all have the same object-set, S , say. Then on homs, (3) gives, for all $a, b \in S$,

$$U_{n+1}(X_{n+1}(a, b)) = X_n(a, b).$$

So X consists of a set S together with, for each $a, b \in S$, a sequence

$$(\dots, X_2(a, b), X_1(a, b), X_0(a, b)) \in \infty\text{-}\mathbf{Gph}.$$

In other words, X is an object of $(\infty\text{-}\mathbf{Gph})\text{-}\mathbf{Gph}$. A similar argument applies to morphisms.

We have now shown that the limit of the original diagram is $\infty\text{-}\mathbf{Gph}$ and that it is preserved by F_G , so by Adámek's Theorem $\infty\text{-}\mathbf{Gph}$ is the terminal coalgebra for F_G as required. ■

Note that by Lambek's Lemma (1.4) we then have:

2.5. COROLLARY. *There is a canonical isomorphism $(\infty\text{-}\mathbf{Gph})\text{-}\mathbf{Gph} \cong \infty\text{-}\mathbf{Gph}$.* □

Finally we discuss the relationship between n -graphs and n -globular sets; this is not directly part of our terminal coalgebra narrative, but will later help us deduce that the categories of n -graphs and ∞ -graphs have the limit/colimit properties required.

Recall that an n -globular set is a diagram

$$X_n \xrightleftharpoons[t]{s} X_{n-1} \xrightleftharpoons[t]{s} \cdots \xrightleftharpoons[t]{s} X_0$$

in \mathbf{Set} satisfying $ss = st$ and $ts = tt$ [3]. Write $n\text{-}\mathbf{GSet}$ for the category of n -globular sets. Clearly $n\text{-}\mathbf{GSet}$ is the category of presheaves on a finite category

$$x_n \xleftarrow{\quad} x_{n-1} \xleftarrow{\quad} \cdots \xleftarrow{\quad} x_0$$

(with equations dual to those above) and there is an evident truncation functor

$$n\text{-}\mathbf{GSet} \longrightarrow (n-1)\text{-}\mathbf{GSet},$$

for each $n \geq 1$.

2.6. PROPOSITION. *For all $n \geq 0$, there is a canonical equivalence of categories*

$$n\text{-}\mathbf{Gph} \simeq n\text{-}\mathbf{GSet},$$

and these equivalences of categories commute with the truncation functors.

PROOF. See the proof of Proposition 1.4.9 of [15]. ■

By contrast with ∞ -graphs, it is straightforward to make an infinite-dimensional version of n -**GSet** directly; this is the category **GSet** of globular sets.

2.7. PROPOSITION. *There is a canonical equivalence of categories*

$$\infty\text{-Gph} \simeq \mathbf{GSet}.$$

Under this equivalence and those of Proposition 2.6, the truncation functor tr_n corresponds to the evident truncation functor $\mathbf{GSet} \rightarrow n\text{-GSet}$.

PROOF. See, for instance, Section F.2 of [15]. ■

The fact that this is an equivalence and not an isomorphism comes down to a slightly technical but nonetheless crucial point: the comparison functor

$$\infty\text{-Gph} \rightarrow \mathbf{GSet}$$

involves moving from a structure with locally defined sets of k -cells, to one where a single set of cells is given for each dimension. This means making some arbitrary choices of coproducts at each dimension k , and thus the composite

$$\mathbf{GSet} \rightarrow \infty\text{-Gph} \rightarrow \mathbf{GSet}$$

is destined to be only isomorphic, and not equal, to the identity. This means that **GSet** is only equivalent to a limit of the sequential diagram, and is not itself a limit. As a consequence, **GSet** is *not* a terminal coalgebra for F_G .

One remedy would be to view **CAT** as a 2-category and consider weak 2-dimensional limits, but we have made the choice in this paper to stick to 1-categorical endofunctors and their terminal coalgebras, constructed as 1-dimensional limits. Thus terminal coalgebras are unique up to isomorphism, not just equivalence. For this reason, while it is often standard to use n -graphs and n -globular sets interchangeably, we shall take care not to do so in this work.

This completes our analysis of $\infty\text{-Gph}$; we are now ready to study ∞ -categories.

3. The category of strict ∞ -categories

In this section we will study the construction of strict ∞ -categories via limits, and hence via terminal coalgebras. Most of the material in this section is not new, but we give it the emphasis we need in order to motivate and illuminate the generalisation to the weak case.

The basic idea is that we have for each $n \in \mathbb{N} \cup \{\infty\}$ a category **Str- n -Cat** of strict n -categories and strict n -functors. Then we have the following commuting diagram of truncation functors, building on the diagram for graphs given just after Definition 2.3

$$\begin{array}{ccccccc}
 & & \mathbf{Str}\text{-}\infty\text{-Cat} & & & & \\
 & \swarrow & \searrow & \searrow & \searrow & & \\
 \cdots & \longrightarrow & \mathbf{Str}\text{-}n\text{-Cat} & \longrightarrow & \mathbf{Str}\text{-}(n-1)\text{-Cat} & \longrightarrow & \cdots \longrightarrow \mathbf{Str}\text{-}0\text{-Cat}
 \end{array}$$

Moreover, this is a limit cone in **Cat**.

The precise story is a little more subtle, largely because we must take care over the difference between n -globular sets and n -graphs, as in the previous section. Thus we have the following two distinct (but equivalent) ways of defining strict ∞ -categories.

1. Directly: a strict n -category is an n -globular set equipped with composition and identities satisfying certain axioms.
2. By induction: **Str-0-Cat** = **Set**, and for $n \geq 1$

$$\mathbf{Str}\text{-}(n+1)\text{-Cat} = (\mathbf{Str}\text{-}n\text{-Cat})\text{-Cat}.$$

The resulting categories of strict n -categories are equivalent for all finite n , and we are accustomed to using these definitions somewhat interchangeably. However for the infinite-dimensional case only the first definition works *a priori*; for the version by iterated enrichment, we must *define* **Str- ∞ -Cat** as the above limit, whereas in the first case the fact that it is a limit is a theorem that follows from the direct (non-inductive) definition.

Standard results show that **Str- n -Cat** is monadic over the category n -**GSet** of n -dimensional globular sets or n -**Gph** for each n , and in the next section we perform the above constructions at the level of monads rather than their algebras.

As discussed in the introduction, throughout this section we will need finite products. Recall from Example 1.8 that **CAT_p** denotes the category of locally small categories with finite products, and functors preserving them. Observe that if \mathcal{V} has finite products then \mathcal{V} -**Gph** does too. This extends to the following result.

3.1. LEMMA. F_G restricts to an endofunctor of **CAT_p**. □

We now define the enriched category endofunctor that is the object of study in this section.

3.2. DEFINITION. We write F_C for the endofunctor

$$(\)\text{-Cat}: \mathbf{CAT}_p \longrightarrow \mathbf{CAT}_p$$

described in Example 1.8.

In this section, “ n -category” means *strict* n -category, and similarly n -**Cat** denotes the category of *strict* n -categories, defined iteratively as follows; this is analogous to our iterative definition of ∞ -graphs.

3.3. DEFINITION. For all $n \geq 0$ we define the category n -**Cat** of (small, strict) n -categories as follows:

- $0\text{-Cat} = \mathbf{Set}$, and
- for all $n \geq 1$, $n\text{-Cat} = F_C((n-1)\text{-Cat}) = ((n-1)\text{-Cat})\text{-Cat}$.

As before, we also define for all $n \geq 1$ the **truncation** functor $U_n : n\text{-Cat} \rightarrow (n-1)\text{-Cat}$, as follows.

- $U_1 : 1\text{-Cat} \rightarrow \mathbf{Set}$ is the functor that sends a category to its set of objects, and
- for all $n \geq 2$, $U_n = F_C(U_{n-1}) = (U_{n-1})_*$.

We now give the definition of the category of ∞ -categories as a limit of the n -dimensional versions.

3.4. DEFINITION. The category $\infty\text{-Cat}$ of (small, strict) ∞ -**categories** is the limit of the diagram

$$\cdots \xrightarrow{U_3} 2\text{-Cat} \xrightarrow{U_2} 1\text{-Cat} \xrightarrow{U_1} 0\text{-Cat}$$

in \mathbf{CAT} . By abuse of notation we write $\mathrm{tr}_n : \infty\text{-Cat} \rightarrow n\text{-Cat}$ for the n th projection of this limit cone (as well as the one for ∞ -graphs), and refer to this version of tr_n and U_n as **truncation**.

Thus we may express a strict ∞ -category as an infinite sequence

$$(\dots, X_2, X_1, X_0)$$

where each X_n is in $n\text{-Cat}$ and is the truncation of X_{n+1} .

In the next section we prove that $\infty\text{-Cat}$ is the terminal coalgebra for F_C ; we will deduce it from the analogous result for monads. Given Adámek's Theorem and the definition of $\infty\text{-Cat}$ as a limit, the following lemma provides a strong hint towards the result.

3.5. LEMMA. There are canonical isomorphisms $(I_n)_{n \geq 0}$ such that the following diagram commutes:

$$\begin{array}{ccccccc} \cdots & \xrightarrow{F_C^3!} & F_C^3 \mathbb{1} & \xrightarrow{F_C^2!} & F_C^2 \mathbb{1} & \xrightarrow{F_C!} & F_C \mathbb{1} \\ & & \downarrow \cong I_2 & & \downarrow \cong I_1 & & \downarrow \cong I_0 \\ \cdots & \xrightarrow{U_3} & 2\text{-Cat} & \xrightarrow{U_2} & 1\text{-Cat} & \xrightarrow{U_1} & 0\text{-Cat} \end{array}$$

PROOF. As in the proof of Theorem 2.4. ■

3.6. THEOREM. The category $\infty\text{-Cat}$ of strict ∞ -categories is the terminal coalgebra for F_C .

This can be proved directly in a manner similar to that for ∞ -graphs (by showing that it is a limit in \mathbf{CAT}_p and F_C preserves the limit); however we defer the proof as we will be able to deduce it from results on monads in the next section.

3.7. **REMARK.** The category of n -categories obtained non-inductively (as n -globular sets with extra structure) is equivalent to $n\text{-}\mathbf{Cat}$; see Proposition 1.4.9 of [15], for instance. Likewise the category of ∞ -categories defined as globular sets with extra structure is equivalent to $\infty\text{-}\mathbf{Cat}$; see for instance Proposition 1.4.12 of [15].

Note that as in the previous section, these are equivalences and not isomorphisms, so we should not expect the category of non-inductively defined ∞ -categories to be a terminal coalgebra for F_C .

4. The monad for strict ∞ -categories

It is well known that strict n -categories are the algebras for a certain monad, and in fact there is a more abstract way of performing the constructions of the previous section, using the *monad* for n -categories rather than the *category* of n -categories. This will be useful in Section 5, where we study structures that are *only* defined as algebras for a monad.

In this section we will see that there is a forgetful functor

$$\infty\text{-}\mathbf{Cat} \longrightarrow \infty\text{-}\mathbf{Gph}$$

that it is monadic, that the induced monad is the limit of the monads for n -categories, and hence, that it is the terminal coalgebra for a certain endofunctor.

As discussed in the introduction will need to impose some conditions on the categories and monads involved. To guarantee the existence of a free enriched category monad we demand finite products and small coproducts that interact well, as follows.

4.1. **DEFINITION.** An ***infinitely distributive category*** is a category with finite products and small coproducts in which the former distribute over the latter. We write \mathbf{CAT}_d for the subcategory of \mathbf{CAT} whose objects are the locally small infinitely distributive categories and whose morphisms are functors preserving the finite products and small coproducts.

Note that this property is inherited by enrichment: if \mathcal{V} is infinitely distributive then $\mathcal{V}\text{-}\mathbf{Gph}$ and $\mathcal{V}\text{-}\mathbf{Cat}$ are too. This extends to the following result.

4.2. **LEMMA.** The endofunctors F_G and F_C restrict to endofunctors of \mathbf{CAT}_d . □

We must also make precise some background on monads.

4.3. **DEFINITION.** We write \mathbf{MND} for the category of monads and lax morphisms of monads (as in [20], where lax morphisms of monads are called monad functors).

Thus, an object of \mathbf{MND} is a pair (\mathcal{V}, T) where $\mathcal{V} \in \mathbf{CAT}$ and T is a monad on \mathcal{V} , and a morphism $(\mathcal{V}, T) \longrightarrow (\mathcal{V}', T')$ is a pair (H, θ) where $H: \mathcal{V} \longrightarrow \mathcal{V}'$ is a functor and θ is a natural transformation

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{H} & \mathcal{V}' \\ T \downarrow & \theta \swarrow & \downarrow T' \\ \mathcal{V} & \xrightarrow{H} & \mathcal{V}' \end{array}$$

satisfying the coherence axioms in [20]. Such a morphism is called **weak** if θ is an isomorphism, and we write \mathbf{MND}^{wk} for the subcategory of \mathbf{MND} consisting of all objects but only the weak morphisms; we will need this restricted category for technical reasons later.

We will sometimes use an alternative point of view on monad morphisms. By the results in [20], a lax morphism of monads $(\mathcal{V}, T) \longrightarrow (\mathcal{V}', T')$ can equivalently be described as a pair of functors H, K making the following square commute

$$\begin{array}{ccc} \mathcal{V}^T & \xrightarrow{K} & \mathcal{V}'^{T'} \\ G^T \downarrow & & \downarrow G'^{T'} \\ \mathcal{V} & \xrightarrow{H} & \mathcal{V}' \end{array}$$

where the vertical arrows are the forgetful functors.

4.4. REMARK. Note that the category \mathbf{MND} could be made into a 2-category using monad transformations, but we prefer to stay at the 1-categorical level and use the theory of terminal coalgebras for endofunctors rather than develop the 2-dimensional version of that theory.

4.5. DEFINITION. We write \mathbf{MND}_d for the subcategory of \mathbf{MND} given as follows:

- objects are pairs (\mathcal{V}, T) where $\mathcal{V} \in \mathbf{CAT}_d$ and T is a monad on \mathcal{V} whose functor part preserves small coproducts
- a morphism $(\mathcal{V}, T) \longrightarrow (\mathcal{V}', T')$ is a lax morphism of monads (H, θ) such that the functor H preserves finite products and small coproducts.

Note that for an object (\mathcal{V}, T) of \mathbf{MND}_d , the functor part of T is *not* required to preserve finite products.

We now define the crucial endofunctor on \mathbf{MND}_d ; this can be thought of as the “monad version” of the previous endofunctor $F_C = (\)\text{-Cat}$, in a sense we will make precise using the functor

$$\mathbf{Alg}: \mathbf{MND} \longrightarrow \mathbf{CAT}$$

that sends a monad to its category of algebras. The idea is that given a monad T on \mathcal{V} , our endofunctor produces the monad on $\mathcal{V}\text{-Gph}$ for “categories enriched in T -algebras” induced by the forgetful functor

$$\mathcal{V}^T\text{-Cat} \longrightarrow \mathcal{V}\text{-Gph}.$$

The next two propositions construct this monad.

4.6. PROPOSITION. [Kelly [10, Theorem 23.4]] *Let \mathcal{V} be an infinitely distributive category. Then the forgetful functor*

$$\mathcal{V}\text{-Cat} \longrightarrow \mathcal{V}\text{-Gph}$$

is monadic. The induced monad is the “free \mathcal{V} -category monad” $\mathbf{fc}_{\mathcal{V}}$, which is the identity on underlying sets of objects, and is given on hom-sets as follows: for a \mathcal{V} -graph A and objects $a, a' \in A$,

$$(\mathbf{fc}_{\mathcal{V}}A)(a, a') = \coprod_{k \in \mathbb{N}, a=a_0, a_1, \dots, a_{k-1}, a_k=a'} A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1).$$

Note that in the case $\mathcal{V} = \mathbf{Set}$ the monad $\mathbf{fc}_{\mathcal{V}}$ is the free category monad on \mathbf{Gph} , which we write as \mathbf{fc} .

Now consider a monad T on an infinitely distributive category \mathcal{V} . Since F_G extends to a 2-functor on \mathbf{CAT} (as remarked after Definition 2.1), there is an induced monad $F_G(T) = T_*$ on $F_G(\mathcal{V}) = \mathcal{V}\text{-Gph}$. Hence $\mathcal{V}\text{-Gph}$ carries two monads, $\mathbf{fc}_{\mathcal{V}}$ and T_* .

4.7. PROPOSITION. *Let $(\mathcal{V}, T) \in \mathbf{MND}_d$. Then:*

i) *The diagonal of the commutative square of forgetful functors*

$$\begin{array}{ccc} \mathcal{V}^T\text{-Cat} & \longrightarrow & \mathcal{V}^T\text{-Gph} \\ \downarrow & & \downarrow \\ \mathcal{V}\text{-Cat} & \longrightarrow & \mathcal{V}\text{-Gph} \end{array}$$

is monadic.

ii) *There is a canonical distributive law of T_* over $\mathbf{fc}_{\mathcal{V}}$, and the resulting monad $\mathbf{fc}_{\mathcal{V}} \circ T_*$ on $\mathcal{V}\text{-Gph}$ is the monad induced by this diagonal functor.*

iii) $(\mathcal{V}\text{-Gph}, \mathbf{fc}_{\mathcal{V}} \circ T_*) \in \mathbf{MND}_d$.

PROOF. For parts (i) and (ii) see Proposition F.1.1 of [15]. There it is assumed that \mathcal{V} is a presheaf category, but the proof needs only that \mathcal{V} is infinitely distributive. The last part is straightforward to check. ■

This gives the action of our endofunctor on objects; the full definition is as follows.

4.8. DEFINITION. *We define an endofunctor*

$$F_M: \mathbf{MND}_d \longrightarrow \mathbf{MND}_d$$

as follows.

- *On objects, $F_M(\mathcal{V}, T) = (\mathcal{V}\text{-Gph}, \mathbf{fc}_{\mathcal{V}} \circ T_*)$.*

- On morphisms, a lax morphism of monads

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{H} & \mathcal{V}' \\ T \downarrow & \theta \swarrow & \downarrow T' \\ \mathcal{V} & \xrightarrow{H} & \mathcal{V}' \end{array}$$

is mapped to the composite

$$\begin{array}{ccc} \mathcal{V}\text{-Gph} & \xrightarrow{H_*} & \mathcal{V}'\text{-Gph} \\ T_* \downarrow & \theta_* \swarrow & \downarrow T'_* \\ \mathcal{V}\text{-Gph} & \xrightarrow{H_*} & \mathcal{V}'\text{-Gph} \\ \text{fc}_{\mathcal{V}} \downarrow & \cong \swarrow & \downarrow \text{fc}_{\mathcal{V}'} \\ \mathcal{V}\text{-Gph} & \xrightarrow{H_*} & \mathcal{V}'\text{-Gph} \end{array}$$

where the bottom square is the isomorphism induced by H preserving products and coproducts.

As noted after Definition 4.3, a lax morphism of monads $(\mathcal{V}, T) \longrightarrow (\mathcal{V}', T')$ may be viewed as a commutative square

$$\begin{array}{ccc} \mathcal{V}^T & \xrightarrow{K} & \mathcal{V}'^{T'} \\ G^T \downarrow & & \downarrow G'^{T'} \\ \mathcal{V} & \xrightarrow{H} & \mathcal{V}' \end{array}$$

When monad morphisms are viewed in this way, the image of such a morphism under F_M is the commutative square

$$\begin{array}{ccc} \mathcal{V}^T\text{-Cat} & \xrightarrow{K_*} & \mathcal{V}'^{T'}\text{-Cat} \\ \downarrow & & \downarrow \\ \mathcal{V}\text{-Gph} & \xrightarrow{H_*} & \mathcal{V}'\text{-Gph} \end{array}$$

where the vertical arrows are the forgetful functors; sometimes this point of view will make our constructions easier.

At the end of this section the results we prove about F_M will be transferred to results about F_C via the functor **Alg** as below.

4.9. PROPOSITION. [Street [20, Theorems 1 and 7]]

There is a chain of adjunctions

$$\begin{array}{ccc} & \mathbf{MND} & \\ \text{Und} \swarrow & \uparrow & \searrow \text{Alg} \\ & \mathbf{CAT} & \end{array}$$

where

$$\begin{array}{ccc} \mathbf{Und} & : \mathbf{MND} & \longrightarrow \mathbf{CAT} \\ (\mathcal{V}, T) & \longmapsto & \mathcal{V} \end{array}$$

$$\begin{array}{ccc} \mathbf{Inc} & : \mathbf{CAT} & \longrightarrow \mathbf{MND} \\ \mathcal{V} & \longmapsto & (\mathcal{V}, 1) \end{array}$$

$$\begin{array}{ccc} \mathbf{Alg} & : \mathbf{MND} & \longrightarrow \mathbf{CAT} \\ (\mathcal{V}, T) & \longmapsto & \mathcal{V}^T. \end{array}$$

The proof of the following proposition can be found in the appendix.

4.10. PROPOSITION. *The chain of adjunctions in Proposition 4.9 restricts to a chain of adjunctions*

$$\begin{array}{ccc} & \mathbf{MND}_d & \\ \text{Und} \swarrow & \uparrow & \searrow \text{Alg} \\ & \mathbf{CAT}_d & \end{array}$$

The endofunctor F_M is a “lift” of the endofunctors F_G and F_C , in the following sense; this will enable us to deduce the results about F_C from those about F_M .

4.11. LEMMA. *The squares*

$$\begin{array}{ccc} \mathbf{MND}_d & \xrightarrow{F_M} & \mathbf{MND}_d \\ \text{Und} \downarrow & & \downarrow \text{Und} \\ \mathbf{CAT}_d & \xrightarrow{F_G} & \mathbf{CAT}_d \end{array} \quad \begin{array}{ccc} \mathbf{MND}_d & \xrightarrow{F_M} & \mathbf{MND}_d \\ \text{Alg} \downarrow & & \downarrow \text{Alg} \\ \mathbf{CAT}_d & \xrightarrow{F_C} & \mathbf{CAT}_d \end{array}$$

commute, the first strictly and the second up to a canonical isomorphism.

PROOF. The first assertion is trivial. For the second: by the lower-left leg we have

$$(\mathcal{V}, T) \mapsto \mathcal{V}^T \mapsto \mathcal{V}^T\text{-Cat}.$$

By the upper-right leg we have

$$(\mathcal{V}, T) \mapsto (\mathcal{V}\text{-Gph}, \mathbf{fc}_{\mathcal{V}} \circ T_*) \mapsto \mathcal{V}\text{-Gph}^{\mathbf{fc}_{\mathcal{V}} \circ T_*}.$$

Thus the result follows from Proposition 4.7, part (ii). \blacksquare

We can now use F_M to define the monads for n -categories. This is analogous to our definitions of $n\text{-Gph}$ and $n\text{-Cat}$.

4.12. DEFINITION. We define for each $n \geq 0$ a monad T_n on $n\text{-Gph}$ as follows:

- T_0 is the identity monad on $\mathbf{Set} = 0\text{-Gph}$
- for all $n \geq 1$, $(n\text{-Gph}, T_n) = F_M((n-1)\text{-Gph}, T_{n-1})$.

By Proposition 4.11 and induction, we have

$$\mathbf{Alg}(T_n) \cong n\text{-Cat}$$

for each n . Further, since the monads T_n are in the image of F_M , they all preserve coproducts.

We also define for each $n \geq 1$ a lax morphism of monads

$$(U_n, \gamma_n): (n\text{-Gph}, T_n) \longrightarrow ((n-1)\text{-Gph}, T_{n-1})$$

in \mathbf{MND}_d as follows:

- $(U_1, \gamma_1): (1\text{-Gph}, \mathbf{fc}) \longrightarrow (\mathbf{Set}, 1)$ is the monad morphism corresponding to the objects functor $\mathbf{ob}: \mathbf{Cat} \longrightarrow \mathbf{Set}$.
- For all $n \geq 2$, $(U_n, \gamma_n) = F_M(U_{n-1}, \gamma_{n-1})$.

Note that each $U_n: n\text{-Gph} \longrightarrow (n-1)\text{-Gph}$ is the truncation functor as in Definition 2.2.

Now when morphisms of monads are viewed as commutative squares, the diagram

$$\dots \xrightarrow{(U_3, \gamma_3)} (2\text{-Gph}, T_2) \xrightarrow{(U_2, \gamma_2)} (1\text{-Gph}, T_1) \xrightarrow{(U_1, \gamma_1)} (0\text{-Gph}, T_0) \quad (4)$$

becomes the following commutative diagram

$$\begin{array}{ccccccc} \dots & \xrightarrow{U_3} & 2\text{-Cat} & \xrightarrow{U_2} & 1\text{-Cat} & \xrightarrow{U_1} & 0\text{-Cat} \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{U_3} & 2\text{-Gph} & \xrightarrow{U_2} & 1\text{-Gph} & \xrightarrow{U_1} & 0\text{-Gph} \end{array}$$

Hence there is an induced forgetful functor

$$\infty\text{-Cat} \longrightarrow \infty\text{-Gph}.$$

4.13. LEMMA. *The forgetful functor $\infty\text{-Cat} \rightarrow \infty\text{-Gph}$ is monadic.*

PROOF. See Theorem F.2.2 of [15]. (The result follows from the monadicity of the functors $n\text{-Cat} \rightarrow n\text{-Gph}$, using the fact that the truncations

$$U_n: n\text{-Cat} \rightarrow (n-1)\text{-Cat}$$

are isofibrations in the sense of [12].) ■

4.14. DEFINITION. *Write T_∞ for the induced monad on $\infty\text{-Gph}$; thus, T_∞ is the monad for ∞ -categories.*

Eventually we will take the limit of the above sequential diagram (4) and show that it gives the monad T_∞ ; for now we show that we have a cone as below.

$$\begin{array}{ccccccc}
 & & (\infty\text{-Gph}, T_\infty) & & & & \\
 & \swarrow & \searrow & \searrow & \searrow & & \\
 & & (\text{tr}_2, \tau_2) & (\text{tr}_1, \tau_1) & (\text{tr}_0, \tau_0) & & \\
 \cdots & \longrightarrow & (2\text{-Gph}, T_2) & \xrightarrow{(U_2, \gamma_2)} & (1\text{-Gph}, T_1) & \xrightarrow{(U_1, \gamma_1)} & (0\text{-Gph}, T_0)
 \end{array} \tag{5}$$

For each $n \geq 0$, the truncation functors tr_n participate in a commutative square

$$\begin{array}{ccc}
 \infty\text{-Cat} & \xrightarrow{\text{tr}_n} & n\text{-Cat} \\
 \downarrow & & \downarrow \\
 \infty\text{-Gph} & \xrightarrow{\text{tr}_n} & n\text{-Gph}
 \end{array}$$

This gives the graph truncation functor tr_n the structure of a lax morphism of monads, which we write as

$$(\text{tr}_n, \tau_n) : (\infty\text{-Gph}, T_\infty) \rightarrow (n\text{-Gph}, T_n).$$

We now embark on the proof that the free ∞ -category monad $(\infty\text{-Gph}, T_\infty)$ is the terminal coalgebra for F_M .

4.15. THEOREM. *The strict ∞ -category monad $(\infty\text{-Gph}, T_\infty)$ is the terminal coalgebra for F_M .*

PROOF. We use Adámek's theorem. We need to consider the limit in \mathbf{MND}_d of

$$\cdots \xrightarrow{F_G^3!} F_M^3 \mathbb{1} \xrightarrow{F_G^2!} F_M^2 \mathbb{1} \xrightarrow{F_G!} F_M \mathbb{1} \tag{6}$$

where $\mathbb{1}$ is the terminal object $(\mathbb{1}, 1)$ of \mathbf{MND}_d . (We also continue to write the terminal object of \mathbf{CAT} as $\mathbb{1}$, and the identity monad as 1 .) We need to show

- i) the limit is $(\infty\text{-}\mathbf{Gph}, T_\infty)$, and
- ii) the limit is preserved by F_M .

First we note that the above diagram (6) is isomorphic to our diagram of n -dimensional monads and truncations; formally, there are canonical isomorphisms $(I_n, \iota_n)_{n \geq 0}$ of monads, with I_n as in the analogous result for graphs (see proof of Theorem 2.4), such that the following diagram commutes:

$$\begin{array}{ccccccc}
 \cdots & \xrightarrow{F_M^3!} & F_M^3 \mathbb{1} & \xrightarrow{F_M^2!} & F_M^2 \mathbb{1} & \xrightarrow{F_M!} & F_M \mathbb{1} \\
 & & \downarrow \cong (I_2, \iota_2) & & \downarrow \cong (I_1, \iota_1) & & \downarrow \cong (I_0, \iota_0) \\
 \cdots & \xrightarrow{(U_3, \gamma_3)} & (\mathbf{2}\text{-}\mathbf{Gph}, T_2) & \xrightarrow{(U_2, \gamma_2)} & (\mathbf{1}\text{-}\mathbf{Gph}, T_1) & \xrightarrow{(U_1, \gamma_1)} & (\mathbf{0}\text{-}\mathbf{Gph}, T_0)
 \end{array}$$

As before, this is true by a straightforward induction: we have

$$F_M \mathbb{1} = (\mathbb{1}\text{-}\mathbf{Gph}, 1) \cong (\mathbf{0}\text{-}\mathbf{Gph}, 1)$$

and as the rightmost square of the diagram commutes, the whole diagram commutes.

This shows that the limit we need to take is over the following diagram in \mathbf{MND}_d

$$\cdots \xrightarrow{(U_3, \gamma_3)} (\mathbf{2}\text{-}\mathbf{Gph}, T_2) \xrightarrow{(U_2, \gamma_2)} (\mathbf{1}\text{-}\mathbf{Gph}, T_1) \xrightarrow{(U_1, \gamma_0)} (\mathbf{0}\text{-}\mathbf{Gph}, T_0)$$

So our aim is to show that the cone (5) is a limit in \mathbf{MND}_d . We know the following.

- i) The underlying graph part is a limit cone in \mathbf{CAT} (Definition 2.3).
- ii) The underlying graph part is a cone in \mathbf{CAT}_d ; this follows from the equivalences

$$n\text{-}\mathbf{Gph} \simeq n\text{-}\mathbf{GSet}$$

$$\infty\text{-}\mathbf{Gph} \simeq \mathbf{GSet}.$$

- iii) The diagram is in \mathbf{MND}^{wk} . For this we need to show that all the monad morphisms are weak, not just lax. Certainly each (tr_n, τ_n) is weak by the proof of Theorem F.2.2 of [15]. (In fact, this proof shows that if we are willing to replace each T_n by an isomorphic monad, then we may arrange that each γ_n and τ_n is an identity.) For the (U_n, γ_n) it suffices to show that $F_M^n!$ is weak for all n . Now $!$ is certainly weak, and F_M preserves weakness—examining the definition of F_M on morphisms (Definition 4.8) we see that if θ is an isomorphism then θ_* is too, which gives the result.

So by Proposition A.5 our diagram is a limit in \mathbf{MND}_d and also in \mathbf{MND} ; we will use the latter in the next proof.

We now show that F_M preserves this limit, also using Proposition A.5. We have the same three steps as above, for F_M of the limit diagram as follows.

- i) The underlying graph part is a limit cone in **CAT**; this amounts to F_G preserving the underlying limit in **CAT** which we know from Section 3.
- ii) The underlying graph part is also a cone in **CAT_d** as above.
- iii) The diagram is in **MND^{wk}** as we have seen above that F_M preserves weakness.

Hence the new cone is also a limit in **MND_d** and **MND**. ■

We can now deduce that $\infty\text{-Cat}$ is the terminal coalgebra for F_C .

Proof of Theorem 3.6. We use Adámek's Theorem. The isomorphisms of Lemma 3.5 tell us it suffices to show that the following is a limit in **CAT_p**.

$$\begin{array}{c}
 \infty\text{-Cat} \\
 \swarrow \quad \searrow \quad \searrow \quad \searrow \\
 \cdots \longrightarrow n\text{-Cat} \longrightarrow (n-1)\text{-Cat} \longrightarrow \cdots \longrightarrow 0\text{-Cat}
 \end{array} \tag{7}$$

and that F_C preserves this limit. We know the following facts.

- i) It is a limit in **CAT** (Definition 3.4).
- ii) It is the image under **Alg** of a cone (5) in **MND_d**.
- iii) **Alg** restricts to a functor **MND_d** \longrightarrow **CAT_d**, so the diagram (7) is a cone in **CAT_d** and hence **CAT_p** (as **CAT_d** is a subcategory of **CAT_p**).
- iv) The inclusion **CAT_p** \hookrightarrow **CAT** reflects limits (Proposition A.3) so the diagram is a limit in **CAT_p**.

To show that $F_C: \mathbf{CAT}_p \longrightarrow \mathbf{CAT}_p$ preserves this limit, it is again enough to show that F_C of the limit cone is a limit cone in **CAT**, as the inclusion reflects limits. We use the commutative square

$$\begin{array}{ccc}
 \mathbf{MND}_d & \xrightarrow{F_M} & \mathbf{MND}_d \\
 \text{Alg} \downarrow & & \downarrow \text{Alg} \\
 \mathbf{CAT}_d & \xrightarrow{F_C} & \mathbf{CAT}_d
 \end{array}$$

of Lemma 4.11. Now the limit cone we need to preserve is diagram (7), which we write as follows

$$(\infty\text{-Cat} \xrightarrow{\text{tr}_n} n\text{-Cat})_{n \geq 0}.$$

This is in fact

$$\text{Alg} \left((\infty\text{-Gph}, T_\infty) \xrightarrow{(\text{tr}_n, \tau_n)} (n\text{-Gph}, T_n) \right)_{n \geq 0}$$

so applying F_C we get

$$F_C \mathbf{Alg} \left((\infty\text{-}\mathbf{Gph}, T_\infty) \xrightarrow{(\mathrm{tr}_n, \tau_n)} (n\text{-}\mathbf{Gph}, T_n) \right)_{n \geq 0}$$

which, by the above commutative square is

$$\mathbf{Alg} F_M \left((\infty\text{-}\mathbf{Gph}, T_\infty) \xrightarrow{(\mathrm{tr}_n, \tau_n)} (n\text{-}\mathbf{Gph}, T_n) \right)_{n \geq 0}$$

and we know this is a limit as follows. In the previous proof we showed that

$$F_M \left((\infty\text{-}\mathbf{Gph}, T_\infty) \xrightarrow{(\mathrm{tr}_n, \tau_n)} (n\text{-}\mathbf{Gph}, T_n) \right)_{n \geq 0}$$

is a limit in \mathbf{MND} ; furthermore we know that \mathbf{Alg} has a left adjoint and so preserves limits (Proposition 4.9). Thus we have the limit required. \square

4.16. **REMARK.** Our proof that $\infty\text{-}\mathbf{Cat}$ is monadic over $\infty\text{-}\mathbf{Gph}$ did not reveal an explicit description of the monad. But in fact this monad, the free strict ∞ -category monad T_∞ , does have an explicit description, at least if one changes $\infty\text{-}\mathbf{Gph}$ to the equivalent category \mathbf{GSet} . Indeed, re-using the notation T_∞ and T_n for the free ∞ - and n -category monads, one may find in the paper [3] of Batanin an explicit construction of the whole limit cone

$$((\mathbf{GSet}, T_\infty) \longrightarrow (n\text{-}\mathbf{GSet}, T_n))_{n \geq 0}.$$

Transferring across the equivalence between globular sets and graphs gives the limit cone

$$\left((\infty\text{-}\mathbf{Gph}, T_\infty) \xrightarrow{(\mathrm{tr}_n, \tau_n)} (n\text{-}\mathbf{Gph}, T_n) \right)_{n \geq 0}.$$

5. Batanin–Leinster weak ∞ -categories

In the case of weak ∞ -categories we cannot simply use a limit of the categories of n -categories as above. This is because truncating a weak ∞ -category to n dimensions does *not* produce a weak n -category, as the top dimension will not be coherent. The main idea of this section is that we must instead make the construction using “incoherent n -categories”—these should not be completely incoherent, but just as incoherent as a truncated weak ∞ -category.

The notion of weak ∞ -category that we use here is Leinster’s, which is based in turn on Batanin’s [3]. See [14] for the definition in concise terms, and [9] or [15] for explanation.

In later sections we will use the same idea to construct a theory of Trimble-like weak ∞ -categories, the point being that while we cannot use Trimble’s inductive definition to produce ∞ -categories, we can use it to produce incoherent n -categories for each finite n , and then construct ∞ -categories as a limit of those.

There are no terminal coalgebras in this section. This is because in the Batanin–Leinster version, the theory of incoherent n -categories is not obtained from the theory

of incoherent $(n - 1)$ -categories by applying an endofunctor; at least, that is not how *we* obtain it. The main results will, however, concern sequential limits of the same type that we have seen repeatedly in applying Adámek’s Theorem, and that we will see again when we come to Trimble n -categories. Thus this section can be seen as motivation and justification for the use of incoherent n -categories in the Trimble case.

First we give a low-dimensional example to illuminate the idea of incoherent n -categories.

5.1. **EXAMPLE.** Let A be a weak ∞ -category. If we truncate its underlying globular set to 2 dimensions, we have the following data: a 2-globular set

$$A_2 \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} A_1 \begin{array}{c} \xrightarrow{s} \\ \xrightarrow{t} \end{array} A_0$$

equipped with identities, composition and coherence cells as for a bicategory, but *satisfying no axioms*. For example we would have specified 2-cells

$$a: (hg)f \Longrightarrow h(gf)$$

and

$$a^*: h(gf) \Longrightarrow (hg)f$$

but no stipulation that these be inverses or satisfy the pentagon axiom. In this way the structure is “not coherent” (since it does not obey the usual coherence laws) but is also “not completely incoherent” since it does give *some* relationship between different composites of the same diagram, just not a strong enough one.

At the heart of this section is the thought that the following two structures are the same even in the case of weak structures:

- an ∞ -category truncated to n dimensions, and
- a k -category truncated to n dimensions, for any $k > n$.

This is stated precisely after Corollary 5.11.

In this section we write $\infty\text{-Cat}$ and $n\text{-Cat}$ for the categories of *weak* ∞ - and n -categories according to the Batanin–Leinster definition. We will recall the definitions shortly, and we will also define the category $n\text{-iCat}$ of incoherent n -categories. Theorem 5.13 states that weak ∞ -categories can be built from incoherent n -categories in the sense that $\infty\text{-Cat}$ is the limit of the following diagram:

$$\cdots \longrightarrow 2\text{-iCat} \longrightarrow 1\text{-iCat} \longrightarrow 0\text{-iCat}$$

We will make the definition of incoherent n -category directly, using only n -dimensional notions, and then show that it is the same as the structure obtained by truncating weak ∞ -categories.

We now recall some definitions. Batanin–Leinster ∞ -categories are defined via T -operads for a particular cartesian monad T . Recall that a cartesian monad is one whose functor part preserves pullbacks and whose natural transformation parts are cartesian, that is, their naturality squares are all pullbacks. In what follows, \mathcal{V} is a category with finite limits and T is a cartesian monad on \mathcal{V} .

5.2. DEFINITION. A ***T -operad*** is a cartesian monad P on \mathcal{V} together with a cartesian natural transformation $\pi: P \Rightarrow T$ commuting with the monad structures. By the standard abuse of notation, we usually refer to a T -operad (P, π) as P .

The history of this definition goes back to a 1971 paper of Burroni [5] though the word “operad” comes from May [17]. An alternative point of view on the definition, closer to Burroni’s and put forward in the work of Batanin [3], is as follows.

The category $T\text{-}\mathbf{Coll}$ of ***T -collections*** is defined to be the slice category $\mathcal{V}/T1$. This carries a natural monoidal structure (as described in [14], for instance). A T -operad is exactly a monoid in the monoidal category $T\text{-}\mathbf{Coll}$. See Corollary 6.2.4 of [15] for a proof, and Chapter 4 of [15] for general explanation and examples of T -operads.

In this paper we will not need the details of this alternative point of view. All we need to know is how a T -operad in the sense of Definition 5.2 gives rise to a collection, namely: for a T -operad (P, π) , the resulting collection is

$$\begin{array}{c} P1 \\ \downarrow \pi_1 \\ T1 \end{array}$$

Let us now investigate what happens as the monad T is varied. This will be applied when we come to consider n -categories for varying values of n . Details can be found in Section 6.7 of [15].

Let

$$(H, \theta): (\mathcal{V}, T) \longrightarrow (\mathcal{V}', T')$$

be a weak morphism of monads, where the categories \mathcal{V} and \mathcal{V}' have finite limits, the functor H preserves them, and the monads T and T' are cartesian. Then any T -operad P gives rise canonically to a T' -operad HP . (This is easier to see in the collection point of view on operads.) Moreover, there is an induced weak morphism of monads

$$(H, \tilde{\theta}): (\mathcal{V}, P) \longrightarrow (\mathcal{V}', HP).$$

We will use these constructions shortly.

We will always take \mathcal{V} to be the category of n -globular sets, and T to be the free strict n -category monad, for varying values of $n \in \mathbb{N} \cup \{\infty\}$. (In this section only, we work with n -globular sets rather than n -graphs.) This gives the notion of globular operad, introduced by Batanin [3].

5.3. DEFINITION. A ***globular operad*** is a T_∞ -operad, where T_∞ is the free strict ∞ -category monad on \mathbf{GSet} . Similarly, for $n \in \mathbb{N}$, a ***(globular) n -operad*** is a T_n -operad, where T_n is the free strict n -category monad on $n\text{-}\mathbf{GSet}$. An ***algebra*** for a globular operad or n -operad is an algebra for its underlying monad.

The definition makes sense because \mathbf{GSet} and $n\text{-}\mathbf{GSet}$ have finite limits (being presheaf categories) and because the monads T_∞ and T_n are cartesian [15, Section F.2].

Later we will need the following technical fact.

5.4. LEMMA. *Let $n \in \mathbb{N} \cup \{\infty\}$ and let P be an n -operad. Then $(n\text{-}\mathbf{GSet}, P) \in \mathbf{MND}_d$.*

(When $n = \infty$, the notation $n\text{-}\mathbf{GSet}$ means \mathbf{GSet} .)

PROOF. We have to show that $n\text{-}\mathbf{GSet}$ is an infinitely distributive category, and that the functor part of P preserves small coproducts. The first assertion is true since $n\text{-}\mathbf{GSet}$ is a presheaf category. The second is a consequence of the following facts: the free strict n -category functor T_n preserves coproducts [15, Section F.2], there exists a cartesian natural transformation $P \Rightarrow T_n$, and coproducts in a presheaf category are stable under pullback. ■

For each $n \geq 0$ we have a morphism of monads

$$(\mathrm{tr}_n, \tau_n) : (\mathbf{GSet}, T_\infty) \longrightarrow (n\text{-}\mathbf{GSet}, T_n)$$

and moreover it is weak (see proof of Theorem 4.15). So by the remarks above, any globular operad P gives rise to an n -operad $\mathrm{tr}_n P$ and a weak morphism of monads

$$(\mathrm{tr}_n, \tilde{\tau}_n) : (\mathbf{GSet}, P) \longrightarrow (n\text{-}\mathbf{GSet}, \mathrm{tr}_n P). \quad (8)$$

Similarly, for each $n \geq 1$ we have the weak morphism of monads

$$(U_n, \gamma_n) : (n\text{-}\mathbf{GSet}, T_n) \longrightarrow ((n-1)\text{-}\mathbf{GSet}, T_{n-1})$$

so any n -operad Q gives rise to an $(n-1)$ -operad $U_n Q$ and a weak morphism of monads

$$(U_n, \tilde{\gamma}_n) : (n\text{-}\mathbf{GSet}, Q) \longrightarrow ((n-1)\text{-}\mathbf{GSet}, U_n Q). \quad (9)$$

We now come to the definitions of weak ∞ -category, weak n -category, and incoherent n -category. The strategy for the first, due to Batanin [3], is to define a weak ∞ -category as an algebra for a certain globular operad. The operad we use is different from Batanin's; see [14]. To define it, we use the concept of “contraction” (which also has a different meaning here from in Batanin's work).

By **collection** and n -**collection**, we mean T_∞ - and T_n -collection, respectively.

5.5. DEFINITION. *Let*

$$\begin{array}{c} A \\ \downarrow p \\ T_\infty 1 \end{array}$$

*be a collection. For $m \in \mathbb{N}$, a pair a, b of m -cells in A is **parallel** if either $m = 0$ or $m \geq 1$, $s(a) = s(b)$ and $t(a) = t(b)$. A **contraction** on the collection p is a function assigning to each*

- $m \in \mathbb{N}$,
- parallel pair $a, b \in A_m$, and

- $(m+1)$ -cell $y: p(a) \longrightarrow p(b)$ in $T_\infty 1$,

an $(m+1)$ -cell $x: a \longrightarrow b$ in A such that $p(x) = y$.

A (globular) **operad with contraction** is a globular operad equipped with a contraction on its underlying collection.

(This definition can also be phrased in terms of a right lifting property; we leave this as an exercise for the interested reader.)

There is a category **OWC** of operads with contraction, whose morphisms are the morphisms of operads preserving those contractions. This category has an initial object [15, Proposition 9.2.2], whose underlying operad we call P_∞ .

5.6. DEFINITION. A **weak ∞ -category** is an algebra for P_∞ , the initial operad with contraction. We write $\infty\text{-Cat}$ for the category \mathbf{GSet}^{P_∞} of weak ∞ -categories and strict ∞ -functors.

We now turn to finite-dimensional structures. In the definition of weak n -category, the notion of contraction is modified at the top dimension to ensure coherence. For incoherent n -categories, this is exactly the part of the definition that we eliminate. Thus the following definition is in essence just a straightforward truncation of the definition of contraction given above; the “incoherence” is relative to what would be necessary if we were defining weak n -categories.

5.7. DEFINITION. An **incoherent contraction** on an n -collection

$$\begin{array}{c} A \\ \downarrow p \\ T_n 1 \end{array}$$

is defined just as contractions were defined in Definition 5.5, but replacing T_∞ by T_n and “ $m \in \mathbb{N}$ ” by “ $m \in \{0, \dots, n-1\}$ ” throughout. The category $n\text{-iOWC}$ of **n -operads with incoherent contraction** is defined similarly.

The category $n\text{-iOWC}$ has an initial object P_n^i , as we shall see.

5.8. DEFINITION. An **incoherent n -category** is an algebra for P_n^i , the initial n -operad with incoherent contraction. We write $n\text{-iCat}$ for the category $n\text{-GSet}^{P_n^i}$ of incoherent n -categories.

While incoherent n -categories are weak structures, the morphisms in the category $n\text{-iCat}$ should be thought of as *strict n -functors*.

5.9. REMARK. We will not need (coherent) weak n -categories in what follows, but it may clarify matters to consider them briefly now. An incoherent contraction on an n -collection A is a **contraction** if for any parallel n -cells x, x' in A ,

$$p(x) = p(x') \Rightarrow x = x'.$$

So the “incoherence” of an incoherent contraction lies in the fact that given a parallel pair a, b of $(n - 1)$ -cells in A and an n -cell $y : p(a) \longrightarrow p(b)$ in $T_n 1$, there may be *many* n -cells $x : a \longrightarrow b$ lifting y .

The definition of weak n -category is given as for incoherent weak n -category, but with contractions in the place of incoherent contractions. That is, the category of n -operads with contraction has an initial object P_n , and a weak n -category is defined as a P_n -algebra. The operad P_n can be viewed as P_n^i with some of its n -dimensional operations identified, so a weak n -category can be viewed as an incoherent n -category satisfying some n -dimensional equations. The significance of this difference is illustrated in Example 5.1.

Returning to the main development, let us consider how the theory of (incoherent) n -categories varies as n varies in $\mathbb{N} \cup \{\infty\}$. Truncation defines, for each n , functors

$$\mathbf{OWC} \xrightarrow{\mathrm{tr}_n} n\text{-}\mathbf{iOWC}, \quad n\text{-}\mathbf{iOWC} \xrightarrow{U_n} (n - 1)\text{-}\mathbf{iOWC}.$$

So we have, for each n , an operad $\mathrm{tr}_n P_\infty$ with incoherent contraction.

5.10. PROPOSITION. *For each $n \geq 0$, $\mathrm{tr}_n P_\infty$ is an initial object of $n\text{-}\mathbf{iOWC}$.*

PROOF. This follows from Proposition 9.3.7 of [15] (where incoherent contractions are called precontractions). ■

Hence $P_n^i = \mathrm{tr}_n P_\infty$. We then have an $(n - 1)$ -operad $U_n P_n^i$ with incoherent contraction.

5.11. COROLLARY. *For each $n \geq 1$, we have $U_n P_n^i = P_{n-1}^i$.*

PROOF. $U_n P_n^i = U_n \mathrm{tr}_n P_\infty = \mathrm{tr}_{n-1} P_\infty = P_{n-1}^i$. ■

It can also be shown that $U_n P_n = P_{n-1}^i$: the theory of weak n -categories truncated to dimension $(n - 1)$ is the theory of incoherent $(n - 1)$ -categories [15, Corollary 9.3.10]. Hence $\mathrm{tr}_n P_\infty = U_{n+1} P_{n+1}$, and, more generally, $\mathrm{tr}_n P_\infty = U_{n+1} U_{n+2} \cdots U_k P_k$ for any $k > n$: the theory of weak ∞ -categories truncated to dimension n is the same as the theory of weak k -categories truncated to dimension n .

We now exhibit $\infty\text{-}\mathbf{Cat}$ as the limit of the categories $n\text{-}\mathbf{iCat}$. For each $n \geq 0$, we have a weak morphism of monads

$$(\mathrm{tr}_n, \tilde{\tau}_n) : (\mathbf{GSet}, P_\infty) \longrightarrow (n\text{-}\mathbf{GSet}, \mathrm{tr}_n P_\infty = P_n^i)$$

by (8). We also have, for each $n \geq 1$, a weak morphism of monads

$$(U_n, \tilde{\gamma}_n) : (n\text{-}\mathbf{GSet}, P_n^i) \longrightarrow ((n - 1)\text{-}\mathbf{GSet}, U_n P_n^i = P_{n-1}^i)$$

by (9). Together these form a cone

$$\begin{array}{c} (\mathbf{GSet}, P_\infty) \\ \swarrow \quad \downarrow \quad \searrow \quad \searrow \\ \cdots \rightarrow (n\text{-}\mathbf{GSet}, P_n^i) \rightarrow ((n - 1)\text{-}\mathbf{GSet}, P_{n-1}^i) \rightarrow \cdots \rightarrow (0\text{-}\mathbf{GSet}, P_0^i) \end{array} \quad (10)$$

in \mathbf{MND} . In fact it is a cone in \mathbf{MND}_d , by Lemma 5.4. Applying the functor \mathbf{Alg} gives a cone

$$\begin{array}{c}
 \infty\text{-}\mathbf{Cat} \\
 \swarrow \quad \downarrow \quad \searrow \quad \searrow \\
 \cdots \longrightarrow n\text{-}\mathbf{iCat} \longrightarrow (n-1)\text{-}\mathbf{iCat} \longrightarrow \cdots \longrightarrow 0\text{-}\mathbf{iCat}
 \end{array} \tag{11}$$

in \mathbf{CAT}_d .

We now prove the main results of this section.

5.12. THEOREM. *The cone (10) is a limit cone in both \mathbf{MND} and \mathbf{MND}_d .*

That is, the monad for weak ∞ -categories is the limit of the monads for incoherent n -categories. This is a precise expression of the idea that the theory of weak ∞ -categories is built up dimension by dimension from the theories of incoherent n -categories.

PROOF. All the arrows in this cone are weak morphisms of monads, and the underlying cone in \mathbf{CAT} is a limit cone. So by Proposition A.5, the cone (10) is a limit in \mathbf{MND} and \mathbf{MND}_d . ■

5.13. THEOREM. *The cone (11) is a limit cone in \mathbf{CAT} , \mathbf{CAT}_p and \mathbf{CAT}_d .*

That is, the category of weak ∞ -categories is the limit of the categories of incoherent n -categories.

PROOF. By Proposition 4.9, the functor $\mathbf{Alg}:\mathbf{MND} \rightarrow \mathbf{CAT}$ preserves limits, so Theorem 5.12 implies that (11) is a limit cone in \mathbf{CAT} . Then by Proposition A.3, it is also a limit cone in \mathbf{CAT}_p and \mathbf{CAT}_d . ■

In the remaining sections we show how to copy this construction to produce a definition of weak ∞ -category for Trimble's theory. As noted in the introduction, a similar point of view on Trimble weak n -categories (for finite n) has been developed independently in work of Weber [22].

6. Trimble ∞ -categories: underlying data

We now embark on our analysis of Trimble's weak n -categories. The constructions in the next three sections resemble those in Sections 2, 3 and 4 but with the functor from \mathbf{Top} built in as discussed in the introduction. We begin by incorporating this data into the “enriched graph” endofunctor, before proceeding to study the “weakly enriched category” endofunctor in the next section, and finally the monad version in Section 8.

Recall that previously we used the endofunctor

$$F_G: \mathcal{V} \mapsto \mathcal{V}\text{-}\mathbf{Gph}.$$

Now we need an endofunctor of the form

$$(\mathbf{Top} \longrightarrow \mathcal{V}) \longmapsto (\mathbf{Top} \longrightarrow \mathcal{V}\text{-}\mathbf{Gph}).$$

This motivates us to work in the slice category $\mathbf{Top}/\mathbf{CAT}$. We now proceed to define the endofunctor we need, and show that its terminal coalgebra is the underlying ∞ -graph functor

$$\mathbf{Top} \xrightarrow{\Pi_\infty} \infty\text{-}\mathbf{Gph}.$$

Let \mathbf{Top} be the category of all topological spaces and continuous maps. The crucial property of \mathbf{Top} here is that it admits the following canonical functor; this is an unbased analogue of the loop space functor Ω .

6.1. DEFINITION. *There is a canonical functor*

$$\Gamma: \mathbf{Top} \longrightarrow \mathbf{Top}\text{-}\mathbf{Gph}$$

defined as follows. Given a space X , the \mathbf{Top} -graph ΓX is given by:

- $\mathbf{ob}(\Gamma X)$ *is the set of points of X*
- *for points x, y of X , the space $(\Gamma X)(x, y)$ is the path space $X(x, y)$.*

Note that paths are always parametrised by $[0, 1]$ so that $X(x, y)$ is naturally a subspace of $X^{[0, 1]}$.

We are now ready to define our endofunctor.

6.2. DEFINITION. *We define an endofunctor*

$$D_G: \mathbf{Top}/\mathbf{CAT} \longrightarrow \mathbf{Top}/\mathbf{CAT}$$

by

$$D_G \left(\mathbf{Top} \xrightarrow{\Pi} \mathcal{V} \right) = \left(\mathbf{Top} \xrightarrow{\Pi^+} \mathcal{V}\text{-}\mathbf{Gph} \right)$$

where Π^+ is the composite functor

$$\mathbf{Top} \xrightarrow{\Gamma} \mathbf{Top}\text{-}\mathbf{Gph} \xrightarrow{\Pi_*} \mathcal{V}\text{-}\mathbf{Gph}.$$

Explicitly, given a space X , the graph $\Pi^+ X$ has

- *objects the set of points of X , and*
- *for points x and y , $(\Pi^+ X)(x, y) = \Pi(X(x, y))$*

On morphisms, D_G is defined by

$$\begin{array}{ccc} \mathbf{Top} & \begin{array}{c} \xrightarrow{\Pi} \mathcal{V} \\ \xrightarrow{\Pi'} \mathcal{V}' \end{array} & \\ & \downarrow H & \\ & \mathcal{V}' & \end{array} \quad \longrightarrow \quad \begin{array}{ccc} & & \mathcal{V}\text{-}\mathbf{Gph} \\ & \xrightarrow{\Pi_*} & \downarrow H_* \\ \mathbf{Top} & \xrightarrow{\Gamma} \mathbf{Top}\text{-}\mathbf{Gph} & \\ & \searrow \Pi'_* & \downarrow H_* \\ & & \mathcal{V}'\text{-}\mathbf{Gph} \end{array}$$

Next we define what might be called the “incoherent” or “truncated” fundamental n -graph functor—instead of taking homotopy classes at the top dimension we simply truncate. This will be the n -dimensional part of our eventual terminal coalgebra, that is, the n th term in the sequential limit. As before, we use the superscript “i” to indicate the incoherence.

6.3. DEFINITION. For each $n \geq 0$, define a functor $\Pi_n^i: \mathbf{Top} \rightarrow n\text{-}\mathbf{Gph}$ by:

- $\Pi_0^i: \mathbf{Top} \rightarrow 0\text{-}\mathbf{Gph} = \mathbf{Set}$ is the functor mapping a space to its set of points
- for $n \geq 1$,

$$\left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_n^i \\ n\text{-}\mathbf{Gph} \end{array} \right) = D_G \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_{n-1}^i \\ (n-1)\text{-}\mathbf{Gph} \end{array} \right)$$

Thus $\Pi_n^i(X)$ is the n -graph consisting of the points of X , the paths, the homotopies between paths, the homotopies between those, and so on up to the n th dimension, where we do not take homotopy classes; this corresponds to the n -globular set whose k -cells are continuous maps from the k -ball into X .

Furthermore, for each $n \geq 1$ we have the morphism

$$U_n: \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_n^i \\ n\text{-}\mathbf{Gph} \end{array} \right) \longrightarrow \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_{n-1}^i \\ (n-1)\text{-}\mathbf{Gph} \end{array} \right)$$

in $\mathbf{Top}/\mathbf{CAT}$, where the underlying functor

$$U_n: n\text{-}\mathbf{Gph} \rightarrow (n-1)\text{-}\mathbf{Gph}$$

is the truncation functor as in the graph case (Definition 2.2).

We wish to define the infinite-dimensional version of Π_n^i as a limit of the n -dimensional versions. The following result tells us we can take limits in the slice category $\mathbf{Top}/\mathbf{CAT}$ as we expect.

6.4. PROPOSITION. [16, Exercise V.1.1] Let \mathcal{C} be a category and $C \in \mathcal{C}$. Then the forgetful functor $C/\mathcal{C} \rightarrow \mathcal{C}$ creates limits.

6.5. COROLLARY. *The diagram*

$$\dots \xrightarrow{U_2} \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_1^i \\ 1\text{-}\mathbf{Gph} \end{array} \right) \xrightarrow{U_1} \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_0^i \\ 0\text{-}\mathbf{Gph} \end{array} \right)$$

has a limit in $\mathbf{Top}/\mathbf{CAT}$ of the form

$$\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_\infty \\ \infty\text{-}\mathbf{Gph} \end{array} \quad (12)$$

6.6. DEFINITION. We call Π_∞ the **fundamental ∞ -graph** functor. We write tr_n for the n th projection of this limit, and call it **truncation**.

6.7. REMARK. The commuting triangles

$$\begin{array}{ccc} & \infty\text{-}\mathbf{Gph} & \\ \Pi_\infty \nearrow & & \downarrow \mathrm{tr}_n \\ \mathbf{Top} & & n\text{-}\mathbf{Gph} \\ \Pi_n^i \searrow & & \end{array}$$

tell us how Π_∞ behaves, that is

$$\Pi_n^i(X) = \mathrm{tr}_n(\Pi_\infty(X)).$$

6.8. REMARK. Note that Γ gives \mathbf{Top} the structure of an F_G -coalgebra, and Π_∞ is then the unique map of coalgebras to the terminal coalgebra $\infty\text{-}\mathbf{Gph}$.

We can now show that Π_∞ is a terminal coalgebra as required. We are going to use our previous analysis of F_G to help us, by means of the following two results. First we note that D_G is a “lift” of F_G in the following sense.

6.9. LEMMA. *The square*

$$\begin{array}{ccc} \mathbf{Top}/\mathbf{CAT} & \xrightarrow{D_G} & \mathbf{Top}/\mathbf{CAT} \\ U \downarrow & & \downarrow U \\ \mathbf{CAT} & \xrightarrow{F_G} & \mathbf{CAT} \end{array}$$

commutes, where U is the forgetful functor. □

The following result will tell us that D_G satisfies the hypotheses of Adámek’s Theorem, because F_G does.

6.10. PROPOSITION. Let \mathcal{C} be a category and $C \in \mathcal{C}$. Let D and F be endofunctors such that the square

$$\begin{array}{ccc} C/\mathcal{C} & \xrightarrow{D} & C/\mathcal{C} \\ U \downarrow & & \downarrow U \\ \mathcal{C} & \xrightarrow{F} & \mathcal{C} \end{array}$$

commutes, where U is the forgetful functor. Suppose that F satisfies the hypotheses of Adámek's Theorem (1.5). Then so does D , and the image under U of the terminal D -coalgebra is the terminal F -coalgebra.

PROOF. This follows from Adámek's Theorem and U creating limits (Proposition 6.4). ■

We are now ready to characterise the terminal coalgebra of D_G . Write $\mathbb{1}$ for the terminal object ($\mathbf{Top} \xrightarrow{!} \mathbb{1}$) of $\mathbf{Top}/\mathbf{CAT}$.

6.11. THEOREM. The category of ∞ -graphs together with the fundamental ∞ -graph functor

$$\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_\infty \\ \infty\text{-Gph} \end{array}$$

is the terminal coalgebra for D_G .

PROOF. By Proposition 6.10 and Lemma 6.9, the terminal coalgebra for D_G exists and is constructed as in Adámek's Theorem. That is, we take the limit of the diagram

$$\dots \xrightarrow{D_G^3!} D_G^3 \mathbb{1} \xrightarrow{D_G^2!} D_G^2 \mathbb{1} \xrightarrow{D_G!} D_G \mathbb{1} \xrightarrow{!} \mathbb{1}$$

But there are canonical isomorphisms $(I_n)_{n \geq 0}$ such that the following diagram commutes:

$$\begin{array}{ccccccc} \dots & \xrightarrow{D_G^3!} & D_G^3 \mathbb{1} & \xrightarrow{D_G^2!} & D_G^2 \mathbb{1} & \xrightarrow{D_G!} & D_G \mathbb{1} \\ & & \downarrow \cong I_2 & & \downarrow \cong I_1 & & \downarrow \cong I_0 \\ & & \mathbf{Top} & & \mathbf{Top} & & \mathbf{Top} \\ & & \downarrow \Pi_2^i & & \downarrow \Pi_1^i & & \downarrow \Pi_0^i \\ \dots & \xrightarrow{U_3} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_2^i \\ \mathbf{2-Gph} \end{array} \right) & \xrightarrow{U_2} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_1^i \\ \mathbf{1-Gph} \end{array} \right) & \xrightarrow{U_1} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_0^i \\ \mathbf{0-Gph} \end{array} \right) \end{array}$$

—the isomorphisms I_n are the same as in the case of F_G (Theorem 2.4). But we have already shown that the limit of the lower part of the diagram is

$$\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_\infty \\ \infty\text{-}\mathbf{Gph} \end{array}$$

thus it is the terminal coalgebra of D_G as required. ■

7. The category of Trimble ∞ -categories

In this section we define Trimble n - and ∞ -categories, and begin the process of characterising the category of Trimble ∞ -categories as a terminal coalgebra. The structure of this section follows that of Section 3 (strict ∞ -categories); as with the strict case we will study the monads in the next section, and deduce the result about terminal coalgebras from the more powerful result for monads. The notation $n\text{-}\mathbf{Cat}$ will now always refer to Trimble n -categories, and similarly $\infty\text{-}\mathbf{Cat}$. Operads will be non-symmetric classical operads.

First we recall the notion of “weak enrichment” used in the Trimble theory; the idea is to enrich in \mathcal{V} in a way that is weakened by the action of an operad P (see [7]).

7.1. DEFINITION. *Let \mathcal{V} be a symmetric monoidal category and P an operad in \mathcal{V} . A (\mathcal{V}, P) -category A is given by*

- *a \mathcal{V} -graph A , equipped with*
- *for all $k \geq 0$ and $a_0, \dots, a_k \in \mathbf{ob} A$ a composition morphism*

$$\gamma: P(k) \otimes A(a_{k-1}, a_k) \otimes \cdots \otimes A(a_0, a_1) \longrightarrow A(a_0, a_k)$$

in \mathcal{V} , compatible with the composition and identities of the operad in the usual way (as for algebras). Morphisms are defined in the obvious way, giving a category $(\mathcal{V}, P)\text{-}\mathbf{Cat}$.

7.2. EXAMPLES. The following degenerate examples show how this notion generalises both \mathcal{V} -categories and P -algebras.

- i) If P is the terminal operad then a (\mathcal{V}, P) -category is exactly a \mathcal{V} -category.
- ii) A (\mathcal{V}, P) -category with only one object is exactly a P -algebra.

There is a particular topological operad E for which (\mathbf{Top}, E) -categories are in plentiful supply; this is the operad used by Trimble in his original definition [21, 14]. We will fix this operad and use it for the rest of the work.

7.3. DEFINITION. We define the operad E in **Top** by setting $E(k)$ to be the space of continuous endpoint-preserving maps

$$[0, 1] \longrightarrow [0, k]$$

for each $k \geq 0$. The composition maps

$$E(m) \times E(k_1) \times \cdots \times E(k_m) \longrightarrow E(k_1 + \cdots + k_m)$$

are given by reparametrisation and the unit is given by the identity map $[0, 1] \longrightarrow [0, 1]$ in $E(1)$.

7.4. EXAMPLE. Every topological space X gives rise canonically to a (\mathbf{Top}, E) -category as follows.

- Its underlying graph is ΓX , defined in Definition 6.1.
- Composition: given points $x_0, \dots, x_k \in X$ we have a canonical map

$$E(k) \times X(x_{k-1}, x_k) \times \cdots \times X(x_0, x_1) \longrightarrow X(x_0, x_k)$$

compatible with the operad composition: given an element of the domain space, we concatenate the k paths and apply the reparametrisation specified by the element of $E(k)$, thus obtaining a single path.

By a slight abuse of notation, we refer to the (\mathbf{Top}, E) -category as ΓX , too.

7.5. REMARK. In all that follows, we use this operad E as our starting point. Note that other operads could be used; the only property of E that is required to make the constructions work is that it “acts on path spaces” as in Example 7.4. In fact, E is in a precise sense the *universal* operad acting on path spaces [6]; a smaller but somewhat less elegant operad is exhibited in [8].

Note, further, that for each $k \geq 0$, the space $E(k)$ is contractible. This is what will give *coherence* for the n -categories we define; but from a technical point of view the induction will not depend on this property of E .

Our aim is to combine

- the move from \mathcal{V} -graphs to \mathcal{V} -categories, as in the move from Section 2 to Section 3, and
- the move from **CAT** to the slice **Top/CAT**, as in the move from Section 2 to Section 6.

That is we wish define an endofunctor of the form

$$(\mathbf{Top} \longrightarrow \mathcal{V}) \longmapsto (\mathbf{Top} \longrightarrow (\mathcal{V}, P)\text{-}\mathbf{Cat})$$

where P is an operad suitably derived from the starting data. As in Section 3 we must restrict to categories with finite products, so now our base category is the slice

$$\mathbf{Top}/\mathbf{CAT}_p.$$

We now define the endofunctor for which the category of Trimble ∞ -categories will be the terminal coalgebra. We proceed in an exactly analogous manner to Section 3, defining an endofunctor for “enrichment”, except now instead of producing ordinary enriched categories, we produce enriched categories weakened by the action of an operad. We begin by giving the direct explicit definition, with a more abstract characterisation afterwards.

7.6. DEFINITION. *We define an endofunctor*

$$D_C: \mathbf{Top}/\mathbf{CAT}_p \longrightarrow \mathbf{Top}/\mathbf{CAT}_p$$

by

$$D_C \left(\mathbf{Top} \xrightarrow{\Pi} \mathcal{V} \right) = \left(\mathbf{Top} \xrightarrow{\Pi^+} (\mathcal{V}, \Pi E)\text{-}\mathbf{Cat} \right)$$

where Π^+ is given as follows. For a space X , the $(\mathcal{V}, \Pi E)$ -category $\Pi^+ X$ has as its underlying \mathcal{V} -graph what was called $\Pi^+ X$ in Section 6, and the action of the operad ΠE is given by

$$\begin{array}{c} \Pi(E(k)) \times \Pi(X(x_{k-1}, x_k)) \times \cdots \times \Pi(X(x_0, x_1)) \\ \cong \downarrow \quad \Pi \text{ preserves products} \\ \Pi(E(k) \times X(x_{k-1}, x_k) \times \cdots \times X(x_0, x_1)) \\ \downarrow \quad \Pi \text{ of the action of } E \text{ on path spaces} \\ \Pi(X(x_0, x_k)). \end{array}$$

Note that here ΠE is the operad defined by $(\Pi E)(k) = \Pi(E(k))$; this is an operad as Π preserves products. It is straightforward to show that if Π preserves products then Π^+ also does.

We now give a more abstract description of this endofunctor, in the style of the previous section. Recall that we defined $D_G(\mathbf{Top} \xrightarrow{\Pi} \mathcal{V})$ as a composite

$$\mathbf{Top} \xrightarrow{\Gamma} \mathbf{Top}\text{-}\mathbf{Gph} \xrightarrow{\Pi_*} \mathcal{V}\text{-}\mathbf{Gph}.$$

Similarly, we will express $D_C(\mathbf{Top} \xrightarrow{\Pi} \mathcal{V})$ as

$$\mathbf{Top} \xrightarrow{\Gamma} (\mathbf{Top}, E)\text{-}\mathbf{Cat} \xrightarrow{\Pi_*} (\mathcal{V}, \Pi E)\text{-}\mathbf{Cat}.$$

By a mild abuse of notation we will generalise the functors Γ and $(\)_*$ of the previous section to the case of categories, without changing the notation. We begin with Γ .

7.7. LEMMA. The assignment $X \mapsto \Gamma X$ of Example 7.4 extends to a functor

$$\Gamma: \mathbf{Top} \longrightarrow (\mathbf{Top}, E)\text{-}\mathbf{Cat},$$

and the following triangle commutes

$$\begin{array}{ccc} & & (\mathbf{Top}, E)\text{-}\mathbf{Cat} \\ & \nearrow \Gamma & \downarrow \\ \mathbf{Top} & & \\ & \searrow \Gamma & \downarrow \\ & & \mathbf{Top}\text{-}\mathbf{Gph} \end{array} \quad (13)$$

where the vertical arrow is the forgetful functor.

We now turn our attention to the functor $(\)_*$. Straightforward calculations give the following.

7.8. LEMMA. Consider $H: \mathcal{V} \longrightarrow \mathcal{W}$ in \mathbf{CAT}_p and an operad P in \mathcal{V} . Then there are an induced operad HP in \mathcal{W} given by $(HP)(k) = H(P(k))$ and an induced functor

$$H_*: (\mathcal{V}, P)\text{-}\mathbf{Cat} \longrightarrow (\mathcal{W}, HP)\text{-}\mathbf{Cat}.$$

Furthermore H_* preserves finite products, that is, lies in \mathbf{CAT}_p .

The endofunctor D_C of $\mathbf{Top}/\mathbf{CAT}_p$ can now be described abstractly by

$$D_C \left(\mathbf{Top} \xrightarrow{\Pi} \mathcal{V} \right) = \left(\mathbf{Top} \xrightarrow{\Pi^+} (\mathcal{V}, \Pi E)\text{-}\mathbf{Cat} \right)$$

where Π^+ is the composite functor

$$\mathbf{Top} \xrightarrow{\Gamma} (\mathbf{Top}, E)\text{-}\mathbf{Cat} \xrightarrow{\Pi_*} (\mathcal{V}, \Pi E)\text{-}\mathbf{Cat}$$

(which is in \mathbf{CAT}_p by the preceding results).

We now give Trimble's definition of n -category inductively by applying D_C repeatedly.

7.9. DEFINITION. For each $n \geq 0$, we define simultaneously

- a category $n\text{-}\mathbf{Cat}$ with finite products, whose objects are called **Trimble n -categories** (but whose morphisms should be thought of as strict n -functors)
- a finite product preserving functor $\Pi_n: \mathbf{Top} \longrightarrow n\text{-}\mathbf{Cat}$, the **fundamental n -groupoid** functor.

We use the fundamental $(n-1)$ -groupoid of each space $E(k)$ to parametrise k -ary composition in an n -category. The definitions are:

- $0\text{-}\mathbf{Cat} = \mathbf{Set}$, and $\Pi_0: \mathbf{Top} \longrightarrow \mathbf{Set}$ is the functor sending a space to its set of path components

- for $n \geq 1$,

$$\left(\mathbf{Top} \xrightarrow{\Pi_n} n\text{-}\mathbf{Cat} \right) = D_C \left(\mathbf{Top} \xrightarrow{\Pi_{n-1}} (n-1)\text{-}\mathbf{Cat} \right).$$

Incoherent Trimble n -categories are defined similarly. As before, the key is to find the part of the definition that gives the coherence axioms, and remove it. In this case it comes from defining $\Pi_0 X$ as the set of path components of X —this ensures coherence at the top dimension. So we simply remove this and replace it with the set of points, as follows.

7.10. DEFINITION. For each $n \geq 0$, we define simultaneously

- a category $n\text{-iCat}$ with finite products, whose objects are called **incoherent Trimble n -categories**
- a finite product preserving functor $\Pi_n^i: \mathbf{Top} \rightarrow n\text{-iCat}$

as follows.

- $0\text{-iCat} = \mathbf{Set}$, and $\Pi_0^i: \mathbf{Top} \rightarrow \mathbf{Set}$ is the functor sending a space to its set of points
- for $n \geq 1$,

$$\left(\mathbf{Top} \xrightarrow{\Pi_n^i} n\text{-iCat} \right) = D_C \left(\mathbf{Top} \xrightarrow{\Pi_{n-1}^i} (n-1)\text{-iCat} \right).$$

(The re-use of the notation Π_n^i from Definition 6.3 will be justified shortly.) We also define, for each $n \geq 1$, a morphism

$$U_n: \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_n^i \\ n\text{-iCat} \end{array} \right) \rightarrow \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_{n-1}^i \\ (n-1)\text{-iCat} \end{array} \right)$$

in $\mathbf{Top}/\mathbf{CAT}_p$ as follows:

- $U_1: 1\text{-iCat} = (\mathbf{Set}, \Pi_0^i E)\text{-Cat} \rightarrow \mathbf{Set}$ is the functor that takes the set of objects
- for $n \geq 2$, $U_n = D_C(U_{n-1})$.

In Section 5 we saw that, for one notion of weak higher-dimensional category, the category of weak ∞ -categories is the limit over all n of the categories of incoherent n -categories. Motivated by this, we make the definition of Trimble weak ∞ -categories analogously; in fact we take a limit over all the Π_n^i so that we get a fundamental ∞ -groupoid functor at the same time.

7.11. DEFINITION. *Define*

$$\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_\infty \\ \infty\text{-}\mathbf{Cat} \end{array}$$

to be the limit in $\mathbf{Top}/\mathbf{CAT}$ of the diagram

$$\dots \xrightarrow{U_3} \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_2^i \\ \mathbf{2-iCat} \end{array} \right) \xrightarrow{U_2} \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_1^i \\ \mathbf{1-iCat} \end{array} \right) \xrightarrow{U_1} \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_0^i \\ \mathbf{0-iCat} \end{array} \right) \quad (14)$$

As in the previous section, Proposition 6.4 tells us that this limit exists and $\infty\text{-}\mathbf{Cat}$ is the limit of the categories $n\text{-iCat}$. Objects of $\infty\text{-}\mathbf{Cat}$ are **Trimble ∞ -categories**, and Π_∞ is the **fundamental ∞ -groupoid** functor. The n th projection of the limit is written tr_n and called **truncation**.

7.12. REMARK. As for the underlying graph version (Remark 6.7), the commuting triangles

$$\begin{array}{ccc} & \Pi_\infty & \infty\text{-}\mathbf{Cat} \\ & \nearrow & \downarrow \mathrm{tr}_n \\ \mathbf{Top} & & n\text{-}\mathbf{Cat} \\ & \searrow \Pi_n^i & \end{array}$$

tell us how Π_∞ behaves.

We will prove that $(\mathbf{Top} \xrightarrow{\Pi_\infty} \infty\text{-}\mathbf{Cat})$ is the terminal coalgebra for D_C , using Adámek's Theorem. As in the strict case, we will deduce this from the result for monads in the next section, but the following result gives a strong indication that the result is true. As usual, we compare the sequential diagram whose limit defines our ∞ -categories, with the sequential diagram whose limit appears in Adámek's Theorem.

7.13. LEMMA. *There are canonical isomorphisms $(I_n)_{n \geq 0}$ such that the following diagram commutes:*

$$\begin{array}{ccccccc} \dots & \xrightarrow{D_C^3!} & D_C^3 \mathbb{1} & \xrightarrow{D_C^2!} & D_C^2 \mathbb{1} & \xrightarrow{D_C!} & D_C \mathbb{1} \\ & & \downarrow \cong I_2 & & \downarrow \cong I_1 & & \downarrow \cong I_0 \\ \dots & \xrightarrow{U_3} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_2^i \\ \mathbf{2-iCat} \end{array} \right) & \xrightarrow{U_2} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_1^i \\ \mathbf{1-iCat} \end{array} \right) & \xrightarrow{U_1} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_0^i \\ \mathbf{0-iCat} \end{array} \right) \end{array}$$

PROOF. As usual, it is straightforward to define I_0 and verify the commutativity of the rightmost square, and the rest follows by repeated application of D_C . \blacksquare

7.14. **THEOREM.** *The category of Trimble ∞ -categories together with the fundamental ∞ -groupoid functor*

$$\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_\infty \\ \infty\text{-}\mathbf{Cat} \end{array}$$

is the terminal coalgebra for D_C .

We defer the proof until the end of the next section.

8. The monad for Trimble ∞ -categories

We continue with a process analogous to that for strict ∞ -categories. We will show that there is a monad for Trimble ∞ -categories and that it is the terminal coalgebra for a certain endofunctor.

As in the previous section, we will need to encode the “fundamental n -category” functors as part of this data. The idea is that our endofunctor should act on data such as:

- the category of n -graphs $n\text{-}\mathbf{Gph}$, together with
- the monad for incoherent n -categories T_n^i on it, equipped with
- the fundamental incoherent n -category functor $\mathbf{Top} \rightarrow n\text{-}\mathbf{iCat}$

and produce the $(n + 1)$ -dimensional version. Expressing $n\text{-}\mathbf{iCat}$ as $\mathbf{Alg}(T_n^i)$ leads us to consider the comma category $\mathbf{Top} \downarrow \mathbf{Alg}$, but as in the analysis of the monad for strict ∞ -categories we must restrict to infinitely distributive categories and the appropriate functors. Thus, we work with the following category. Recall from Proposition 4.10 that $\mathbf{Alg} : \mathbf{MND} \rightarrow \mathbf{CAT}$ restricts to a functor $\mathbf{MND}_d \rightarrow \mathbf{CAT}_d$, which we also call \mathbf{Alg} or, for emphasis, \mathbf{Alg}_d .

8.1. **DEFINITION.** *We denote by $\mathbf{Top} \downarrow \mathbf{Alg}_d$ the comma category whose objects are triples (\mathcal{V}, T, Π) where*

- \mathcal{V} is an infinitely distributive category
- T is a monad on \mathcal{V} whose functor part preserves coproducts, and
- Π is a functor $\mathbf{Top} \rightarrow \mathcal{V}^T$ preserving coproducts and finite products.

We often write an object (\mathcal{V}, T, Π) as

$$\left(\mathbf{Top} \xrightarrow{\Pi} \mathcal{V}^T \right).$$

8.2. **EXAMPLE.** For a prototype example of such an object, consider $\mathcal{V} = \mathbf{Gph}$, $T = \mathbf{fc}$, $\Pi =$ the fundamental groupoid functor, that is

$$\Pi: \mathbf{Top} \longrightarrow \mathbf{Gph}^{\mathbf{fc}} = \mathbf{Cat}.$$

Viewing lax morphisms of monads as commutative squares, a morphism

$$(\mathcal{V}, T, \Pi) \longrightarrow (\mathcal{V}', T', \Pi')$$

consists of functors H and K , preserving finite products and small coproducts, such that the following diagram commutes:

$$\begin{array}{ccc} & \mathbf{Top} & \\ \Pi \swarrow & & \searrow \Pi' \\ \mathcal{V}^T & \xrightarrow{K} & \mathcal{V}'^{T'} \\ G^T \downarrow & & \downarrow G^{T'} \\ \mathcal{V} & \xrightarrow{H} & \mathcal{V}' \end{array} \quad (15)$$

Alternatively, it is a morphism $(H, \theta): (\mathcal{V}, T) \longrightarrow (\mathcal{V}', T')$ making the triangle in (15) commute, where K is the induced functor on algebras.

A standard argument proves:

8.3. **LEMMA.** *The adjunction $\mathbf{Inc} \dashv \mathbf{Alg}$ of Proposition 4.10 induces an isomorphism of categories*

$$\mathbf{Top} \downarrow \mathbf{Alg}_d \cong (\mathbf{Top}, 1) / \mathbf{MND}_d.$$

(Here 1 is the identity monad on \mathbf{Top} .)

From this point of view, an object of the category is an infinitely distributive monad (\mathcal{V}, T) together with a lax morphism of monads $(\mathbf{Top}, 1) \longrightarrow (\mathcal{V}, T)$ preserving finite products and small coproducts. A morphism in this category is a commutative triangle. We will use both points of view; the slice category point of view will enable us to calculate certain limits more easily.

Next we define the endofunctor D_M of $\mathbf{Top} \downarrow \mathbf{Alg}_d$, whose terminal coalgebra will turn out to be the monad for Trimble ∞ -categories. First it is useful to note the following result.

8.4. **LEMMA.** *The endofunctors D_G and D_C of $\mathbf{Top}/\mathbf{CAT}$ each restrict to an endofunctor of $\mathbf{Top}/\mathbf{CAT}_d$.*

PROOF. Recall that D_G is defined by

$$D_G(\mathbf{Top} \xrightarrow{\Pi} \mathcal{V}) = (\mathbf{Top} \xrightarrow{\Gamma} \mathbf{Top}\text{-}\mathbf{Gph} \xrightarrow{\Pi_*} \mathcal{V}\text{-}\mathbf{Gph}).$$

Now if \mathcal{V} and Π are in \mathbf{CAT}_d then certainly $\mathcal{V}\text{-}\mathbf{Gph}$ and Π_* are; we need to show that Γ is in \mathbf{CAT}_d , that is, that it commutes with finite products and small coproducts. The

latter is straightforward; the former follows from the definition of the path space $X(x, y)$ as the equaliser of the pair

$$X^{[0,1]} \xrightarrow[(\Delta x, \Delta y)]{(\mathbf{ev}_0, \mathbf{ev}_1)} X \times X$$

where $X^{[0,1]}$ is the exponential in **Top** (which exists since $[0, 1]$ is compact Hausdorff), \mathbf{ev}_0 and \mathbf{ev}_1 are evaluation at 0 and 1, and Δx and Δy are constant at x and y .

We now turn our attention to D_C . Recall that we have the definition

$$D_C \left(\mathbf{Top} \xrightarrow{\Pi} \mathcal{V} \right) = \left(\mathbf{Top} \xrightarrow{\Gamma} (\mathbf{Top}, E)\text{-Cat} \xrightarrow{\Pi_*} (\mathcal{V}, \Pi E)\text{-Cat} \right)$$

with Γ defined in Lemma 7.7. If \mathcal{V} and Π are in **CAT_d** then a straightforward calculation shows that $(\mathcal{V}, \Pi E)\text{-Cat}$ and Π_* are too. We now show that $\Gamma \in \mathbf{CAT}_d$. We use the commuting triangle from Lemma 7.7

$$\begin{array}{ccc} & & (\mathbf{Top}, E)\text{-Cat} \\ & \nearrow \Gamma & \downarrow \\ \mathbf{Top} & & \\ & \searrow \Gamma & \downarrow \\ & & \mathbf{Top}\text{-Gph} \end{array}$$

where the vertical arrow is the forgetful functor. A routine calculation shows that if $\mathcal{V} \in \mathbf{CAT}_d$ then the forgetful functor

$$(\mathcal{V}, P)\text{-Cat} \longrightarrow \mathcal{V}\text{-Gph}$$

creates finite products and small coproducts, and the result then follows from our analysis of $\Gamma : \mathbf{Top} \longrightarrow \mathbf{Top}\text{-Gph}$ above. \blacksquare

We now define the crucial endofunctor on **Top_dAlg**. We proceed analogously to our process of defining F_M , with two propositions analogous to 4.6 and 4.7. The following result generalises Kelly's theorem (4.6) on free enriched categories.

8.5. PROPOSITION. *Let \mathcal{V} be an infinitely distributive category and P an operad in \mathcal{V} . Then the forgetful functor*

$$(\mathcal{V}, P)\text{-Cat} \longrightarrow \mathcal{V}\text{-Gph}$$

is monadic. The induced monad is the “free (\mathcal{V}, P) -category monad” $\mathbf{fc}_{(\mathcal{V}, P)}$, which is the identity on underlying sets of objects, and is given on hom-sets as follows: for a \mathcal{V} -graph A and objects $a, a' \in A$,

$$(\mathbf{fc}_{(\mathcal{V}, P)} A)(a, a') = \coprod_{k \in \mathbb{N}, a = a_0, a_1, \dots, a_{k-1}, a_k = a'} P(k) \times A(a_{k-1}, a_k) \times \cdots \times A(a_0, a_1).$$

PROOF. See Proposition 2.11 of [7]. \blacksquare

Suppose now that we are given an infinitely distributive category \mathcal{V} , an operad P in \mathcal{V}^T , and a coproduct-preserving monad T on \mathcal{V} . Since the forgetful functor

$$G^T: \mathcal{V}^T \longrightarrow \mathcal{V}$$

preserves products, there is an induced operad $G^T P$ in \mathcal{V} , which we shall often simply refer to as P . We then have a functor

$$G_*^T: (\mathcal{V}^T, P)\text{-Cat} \longrightarrow (\mathcal{V}, G^T P)\text{-Cat} = (\mathcal{V}, P)\text{-Cat}.$$

We also have two monads on $\mathcal{V}\text{-Gph}$: the monad $\mathbf{fc}_{(\mathcal{V}, P)}$ of Proposition 8.5, and the monad $T_* = F_G(T)$. The following result generalises the one for ordinary enriched categories (Proposition 4.7).

8.6. PROPOSITION. *Let $(\mathcal{V}, T) \in \mathbf{MND}_d$ and let P be an operad in \mathcal{V}^T . Then:*

i) *The diagonal of the commutative square of forgetful functors*

$$\begin{array}{ccc} (\mathcal{V}^T, P)\text{-Cat} & \longrightarrow & \mathcal{V}^T\text{-Gph} \\ G_*^T \downarrow & & \downarrow G_*^T \\ (\mathcal{V}, P)\text{-Cat} & \longrightarrow & \mathcal{V}\text{-Gph} \end{array}$$

is monadic.

ii) *There is a canonical distributive law of T_* over $\mathbf{fc}_{(\mathcal{V}, P)}$, and the resulting monad $\mathbf{fc}_{(\mathcal{V}, P)} \circ T_*$ on $\mathcal{V}\text{-Gph}$ is the monad induced by this diagonal functor.*

iii) $(\mathcal{V}\text{-Gph}, \mathbf{fc}_{(\mathcal{V}, P)} \circ T_*) \in \mathbf{MND}_d$.

PROOF. Parts (1) and (2) are Theorem 4.1 of [7]. Part (3) is a straightforward calculation (cf. the analogous part of Proposition 4.7). \blacksquare

8.7. DEFINITION. *We define an endofunctor*

$$D_M: \mathbf{Top} \downarrow \mathbf{Alg}_d \longrightarrow \mathbf{Top} \downarrow \mathbf{Alg}_d$$

by

$$D_M(\mathcal{V}, T, \Pi) = (\mathcal{V}\text{-Gph}, T^+, \Pi^+)$$

as follows. Recall that this notation means Π is a functor

$$\mathbf{Top} \xrightarrow{\Pi} \mathcal{V}^T.$$

- T^+ is the monad $\mathbf{fc}_{(\mathcal{V}, \Pi E)} \circ T_*$ on $\mathcal{V}\text{-Gph}$, corresponding to the (monadic) forgetful functor

$$(\mathcal{V}^T, \Pi E)\text{-Cat} \longrightarrow \mathcal{V}\text{-Gph}.$$

Note that as Π preserves products it does indeed induce an operad ΠE in \mathcal{V}^T .

- Π^+ is the functor

$$\mathbf{Top} \xrightarrow{\Gamma} (\mathbf{Top}, E)\text{-Cat} \xrightarrow{\Pi_*} (\mathcal{V}^T, \Pi E)\text{-Cat} \cong \mathcal{V}\text{-Gph}^{T^+}.$$

Note that both $(\mathcal{V}^T, \Pi E)\text{-Cat}$ and Π^+ are in \mathbf{CAT}_d (Lemma 8.4), so $(\mathcal{V}\text{-Gph}, T^+, \Pi^+)$ is indeed an object of $\mathbf{Top} \downarrow \mathbf{Alg}_d$.

We now define D_M on morphisms. Given a morphism

$$(\mathcal{V}, T, \Pi) \longrightarrow (\mathcal{V}', T', \Pi')$$

expressed as

$$\begin{array}{ccc} & \mathbf{Top} & \\ \Pi \swarrow & & \searrow \Pi' \\ \mathcal{V}^T & \xrightarrow{K} & \mathcal{V}'^{T'} \\ G^T \downarrow & & \downarrow G^{T'} \\ \mathcal{V} & \xrightarrow{H} & \mathcal{V}' \end{array} \quad (16)$$

in $\mathbf{Top} \downarrow \mathbf{Alg}_d$, there is an induced morphism

$$(\mathcal{V}\text{-Gph}, T^+, \Pi^+) \longrightarrow (\mathcal{V}'\text{-Gph}, T'^+, \Pi'^+)$$

in $\mathbf{Top} \downarrow \mathbf{Alg}_d$, namely

$$\begin{array}{ccc} & \mathbf{Top} & \\ \Pi^+ \swarrow & & \searrow \Pi'^+ \\ (\mathcal{V}^T, \Pi E)\text{-Cat} & \xrightarrow{K_*} & (\mathcal{V}'^{T'}, \Pi' E)\text{-Cat} \\ \downarrow & & \downarrow \\ \mathcal{V}\text{-Gph} & \xrightarrow{H_*} & \mathcal{V}'\text{-Gph} \end{array}$$

This is a morphism in $\mathbf{Top} \downarrow \mathbf{Alg}_d$ as D_G and D_C both restrict to $\mathbf{Top}/\mathbf{CAT}_d$. This completes the definition of D_M .

8.8. REMARK. We can also directly describe the effect of D_M on the isomorphic category $(\mathbf{Top}, 1)/\mathbf{MND}_d$. The image under D_M of an object

$$(\Lambda, \lambda): (\mathbf{Top}, 1) \longrightarrow (\mathcal{V}, T)$$

is an object of the form

$$(\mathbf{Top}, 1) \longrightarrow (\mathcal{V}\text{-Gph}, T^+)$$

whose underlying functor is $\Lambda^+ : \mathbf{Top} \rightarrow \mathcal{V}\text{-}\mathbf{Gph}$ (as defined in Section 6). The image under D_M of a morphism

$$\begin{array}{ccc} & (\mathbf{Top}, 1) & \\ (\Lambda, \lambda) \swarrow & & \searrow (\Lambda', \lambda') \\ (\mathcal{V}, T) & \xrightarrow{(H, \theta)} & (\mathcal{V}', T') \end{array}$$

is

$$\begin{array}{ccc} & (\mathbf{Top}, 1) & \\ \swarrow & & \searrow \\ (\mathcal{V}\text{-}\mathbf{Gph}, T^+) & \xrightarrow{(H_*, \theta^+)} & (\mathcal{V}'\text{-}\mathbf{Gph}, T'^+) \end{array}$$

where θ^+ is the composite natural transformation

$$\begin{array}{ccc} \mathcal{V}\text{-}\mathbf{Gph} & \xrightarrow{H_*} & \mathcal{V}'\text{-}\mathbf{Gph} \\ \downarrow T_* & \swarrow \theta_* & \downarrow T'_* \\ \mathcal{V}\text{-}\mathbf{Gph} & \xrightarrow{H_*} & \mathcal{V}'\text{-}\mathbf{Gph} \\ \downarrow \mathbf{fc}_{(\mathcal{V}, \Lambda E)} & \swarrow \cong & \downarrow \mathbf{fc}_{(\mathcal{V}', \Lambda' E)} \\ \mathcal{V}\text{-}\mathbf{Gph} & \xrightarrow{H_*} & \mathcal{V}'\text{-}\mathbf{Gph} \end{array} \quad (17)$$

and the bottom square is the isomorphism induced by H preserving products and coproducts.

As in Section 3 we will transfer results about D_M to results about D_C . We now make precise the relationship between those functors. First note that the functors

$$\mathbf{Und}, \mathbf{Alg} : \mathbf{MND}_d \rightarrow \mathbf{CAT}_d$$

of Proposition 4.10 induce functors

$$\mathbf{Und}, \mathbf{Alg} : (\mathbf{Top}, 1)/\mathbf{MND}_d \rightarrow \mathbf{Top}/\mathbf{CAT}_d.$$

8.9. LEMMA. *The squares*

$$\begin{array}{ccc} (\mathbf{Top}, 1)/\mathbf{MND}_d & \xrightarrow{D_M} & (\mathbf{Top}, 1)/\mathbf{MND}_d \\ \downarrow \mathbf{Und} & & \downarrow \mathbf{Und} \\ \mathbf{Top}/\mathbf{CAT}_d & \xrightarrow{D_G} & \mathbf{Top}/\mathbf{CAT}_d \end{array}$$

$$\begin{array}{ccc}
(\mathbf{Top}, 1)/\mathbf{MND}_d & \xrightarrow{D_M} & (\mathbf{Top}, 1)/\mathbf{MND}_d \\
\downarrow \mathbf{Alg} & & \downarrow \mathbf{Alg} \\
\mathbf{Top}/\mathbf{CAT}_d & \xrightarrow{D_C} & \mathbf{Top}/\mathbf{CAT}_d
\end{array}$$

commute, the first strictly and the second up to a canonical isomorphism.

PROOF. Follows from Proposition 8.6: compare the proof of Lemma 4.11. ■

We can now use D_M iteratively to define the monads for incoherent n -categories; this is analogous to Definition 4.12.

8.10. DEFINITION. We define a sequence of objects

$$(n\text{-}\mathbf{Gph}, T_n^i, \Pi_n^i) \in \mathbf{Top} \downarrow \mathbf{Alg}_d$$

as follows.

- T_0^i is the identity monad on $\mathbf{Set} = 0\text{-}\mathbf{Gph}$, and

$$\Pi_0^i: \mathbf{Top} \longrightarrow \mathbf{Set} \cong 0\text{-}\mathbf{Gph}^{T_0^i}$$

is the set-of-points functor, and

- for $n \geq 1$,

$$(n\text{-}\mathbf{Gph}, T_n^i, \Pi_n^i) = D_M((n-1)\text{-}\mathbf{Gph}, T_{n-1}^i, \Pi_{n-1}^i).$$

(We appear to be re-using the notation Π_n^i once again; this will be justified shortly.) We also define, for each $n \geq 1$, a morphism

$$(U_n, \gamma_n): (n\text{-}\mathbf{Gph}, T_n^i, \Pi_n^i) \longrightarrow ((n-1)\text{-}\mathbf{Gph}, T_{n-1}^i, \Pi_{n-1}^i)$$

in $\mathbf{Top} \downarrow \mathbf{Alg}_d$ as follows.

- (U_1, γ_1) is the morphism of monads corresponding to the commutative diagram

$$\begin{array}{ccccc}
& & \mathbf{Top} & & \\
& \swarrow & & \searrow & \\
(\mathbf{Set}, \Pi_0^i E)\text{-}\mathbf{Cat} \cong 1\text{-}\mathbf{Gph}^{T_1^i} & \xrightarrow{\text{ob}} & 0\text{-}\mathbf{Gph}^{T_0^i} \cong \mathbf{Set} & & \\
\downarrow & & \downarrow \cong & & \\
1\text{-}\mathbf{Gph} & \xrightarrow{\text{ob}} & 0\text{-}\mathbf{Gph}; & &
\end{array}$$

note that this is in $\mathbf{Top} \downarrow \mathbf{Alg}_d$, so we may apply D_M .

- For $n \geq 2$, $(U_n, \gamma_n) = D_M(U_{n-1}, \gamma_{n-1})$.

Note that Lemma 8.9 together with induction tells us that

- the category of algebras for each T_n^i is $n\text{-iCat}$, and
- each functor

$$\Pi_n^i: \mathbf{Top} \longrightarrow n\text{-Gph}^{T_n^i} \cong n\text{-iCat}$$

appearing in the sequence of objects $(n\text{-Gph}, T_n^i, \Pi_n^i)$ is the same functor Π_n^i as defined in Section 7.

Furthermore when morphisms of monads are viewed as commutative squares, the diagram

$$\dots \xrightarrow{(U_2, \gamma_2)} (1\text{-Gph}, T_1^i, \Pi_1^i) \xrightarrow{(U_1, \gamma_1)} (0\text{-Gph}, T_0^i, \Pi_0^i) \quad (18)$$

becomes the commutative diagram

$$\begin{array}{ccccc} \dots & \xrightarrow{U_3} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_2^i \\ 2\text{-iCat} \end{array} \right) & \xrightarrow{U_2} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_1^i \\ 1\text{-iCat} \end{array} \right) & \xrightarrow{U_1} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_0^i \\ 0\text{-iCat} \end{array} \right) \\ & & \downarrow & & \downarrow & & \downarrow \\ \dots & \xrightarrow{U_3} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_2^i \\ 2\text{-Gph} \end{array} \right) & \xrightarrow{U_2} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_1^i \\ 1\text{-Gph} \end{array} \right) & \xrightarrow{U_1} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_0^i \\ 0\text{-Gph} \end{array} \right). \end{array} \quad (19)$$

8.11. LEMMA. *Diagram (19) induces a forgetful morphism*

$$\left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_\infty^i \\ \infty\text{-Cat} \end{array} \right) \longrightarrow \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_\infty^i \\ \infty\text{-Gph} \end{array} \right)$$

with underlying functor

$$\infty\text{-Cat} \longrightarrow \infty\text{-Gph}.$$

This functor is monadic.

PROOF. This follows from the fact that each forgetful functor $n\text{-iCat} \longrightarrow n\text{-Gph}$ is monadic, using the same argument as in the proof of Theorem F.2.2 of [15]. ■

8.12. **DEFINITION.** Write T_∞ for the induced monad on $\infty\text{-Gph}$. Thus, T_∞ is the monad for Trimble ∞ -categories, and we have the functor

$$\Pi_\infty: \mathbf{Top} \longrightarrow \infty\text{-Gph}^{T_\infty}.$$

We will show later that T_∞ preserves coproducts and that Π_∞ preserves both coproducts and finite products.

For each $n \geq 0$, the truncation morphisms participate in a commutative square

$$\begin{array}{ccc} \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_\infty \\ \infty\text{-Cat} \end{array} \right) & \xrightarrow{\mathbf{tr}_n} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_n^i \\ n\text{-iCat} \end{array} \right) \\ \downarrow & & \downarrow \\ \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_\infty \\ \infty\text{-Gph} \end{array} \right) & \xrightarrow{\mathbf{tr}_n} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_n^i \\ n\text{-Gph} \end{array} \right) \end{array}$$

This gives graph truncation the structure of a lax morphism of monads, say

$$(\mathbf{tr}_n, \tau_n): (\infty\text{-Gph}, T_\infty) \longrightarrow (n\text{-Gph}, T_n^i),$$

commuting with the morphisms from $(\mathbf{Top}, 1)$.

We now embark on the proof of one of our two main theorems, characterizing the monad T_∞ for Trimble ∞ -categories as the terminal coalgebra for D_M .

8.13. **THEOREM.** The object $(\infty\text{-Gph}, T_\infty, \Pi_\infty)$ of $\mathbf{Top} \downarrow \mathbf{Alg}_d$, consisting of the monad T_∞ for Trimble ∞ -categories together with the fundamental ∞ -groupoid functor Π_∞ , is the terminal coalgebra for D_M .

PROOF. We use Adámek's theorem. We need to consider the limit in $\mathbf{Top} \downarrow \mathbf{Alg}_d$ of

$$\dots \xrightarrow{D_G^3!} D_M^3 \mathbb{1} \xrightarrow{D_G^2!} D_M^2 \mathbb{1} \xrightarrow{D_G!} D_M \mathbb{1} \quad (20)$$

where $\mathbb{1}$ is the terminal object $(\mathbb{1}, 1, !)$ of $\mathbf{Top} \downarrow \mathbf{Alg}_d$. (We also continue to write the terminal object of \mathbf{CAT} as $\mathbb{1}$.) We need to show

- i) the limit is $(\infty\text{-Gph}, T_\infty, \Pi_\infty)$, and
- ii) the limit is preserved by D_M .

First we note that the above diagram (20) is isomorphic to our diagram of n -dimensional monads and truncations; formally, there are canonical isomorphisms $(I_n, \iota_n)_{n \geq 0}$ in $\mathbf{Top} \downarrow \mathbf{Alg}_d$ making the following diagram commute

$$\begin{array}{ccccccc}
 \dots & \xrightarrow{D_M^3!} & D_M^3 \mathbb{1} & \xrightarrow{D_M^2!} & D_M^2 \mathbb{1} & \xrightarrow{D_M!} & D_M \mathbb{1} \\
 & & \downarrow \cong (I_2, \iota_2) & & \downarrow \cong (I_1, \iota_1) & & \downarrow \cong (I_0, \iota_0) \\
 \dots & \xrightarrow{U_3} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_2^i \\ 2\text{-Gph}^{T_2^i} \end{array} \right) & \xrightarrow{U_2} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_1^i \\ 1\text{-Gph}^{T_1^i} \end{array} \right) & \xrightarrow{U_1} & \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_0^i \\ 0\text{-Gph}^{T_0^i} \end{array} \right)
 \end{array}$$

with the underlying functors I_n as in the proof of Theorem 2.4, as usual. Also as usual, this is true by a straightforward induction: we have

$$D_M \mathbb{1} = (\mathbb{1}\text{-Gph}, 1, !) \cong (0\text{-Gph}, 1, !)$$

and as the rightmost square of the diagram commutes, the whole diagram commutes.

This shows that the limit we need to take is over the following diagram in $\mathbf{Top} \downarrow \mathbf{Alg}_d$

$$\dots \xrightarrow{(U_3, \gamma_3)} (2\text{-Gph}, T_2^i, \Pi_2^i) \xrightarrow{(U_2, \gamma_2)} (1\text{-Gph}, T_1^i, \Pi_1^i) \xrightarrow{(U_1, \gamma_0)} (0\text{-Gph}, T_0^i, \Pi_0^i) \quad (21)$$

so our aim is to show that the following cone is a limit in $\mathbf{Top} \downarrow \mathbf{Alg}_d$.

$$\begin{array}{ccccc}
 (\infty\text{-Gph}, T_\infty, \Pi_\infty) & & & & \\
 \swarrow & \searrow & \searrow & \searrow & \\
 & (tr_2, \tau_2) & (tr_1, \tau_1) & (tr_0, \tau_0) & \\
 \dots \longrightarrow & (2\text{-Gph}, T_2^i, \Pi_2^i) \longrightarrow & (1\text{-Gph}, T_1^i, \Pi_1^i) \longrightarrow & (0\text{-Gph}, T_0^i, \Pi_0^i) & \\
 & & & & (22)
 \end{array}$$

So far we only know it is a cone in $\mathbf{Top} \downarrow \mathbf{Alg}$. We make use of the isomorphism

$$\mathbf{Top} \downarrow \mathbf{Alg}_d \cong (\mathbf{Top}, 1) / \mathbf{MND}_d.$$

Now the forgetful functor

$$(\mathbf{Top}, 1) / \mathbf{MND}_d \longrightarrow \mathbf{MND}_d$$

creates limits (Proposition 6.4) so it is enough for us to show

- A. the cone above is in fact in $(\mathbf{Top}, 1) / \mathbf{MND}_d$, not just $(\mathbf{Top}, 1) / \mathbf{MND}$, and
- B. the underlying cone is a limit in \mathbf{MND}_d .

The only complicated aspect of this is checking the correct distributivity/(co)limit preserving conditions. We proceed in steps.

1. First note that by definition the diagram (21) whose limit we are taking is in $\mathbf{Top} \downarrow \mathbf{Alg}_d$, so in particular the underlying diagram

$$\dots \xrightarrow{(U_3, \gamma_3)} (2\text{-}\mathbf{Gph}, T_2^i) \xrightarrow{(U_2, \gamma_2)} (1\text{-}\mathbf{Gph}, T_1^i) \xrightarrow{(U_1, \gamma_1)} (0\text{-}\mathbf{Gph}, T_0^i)$$

is in \mathbf{MND}_d .

2. Also note that the underlying cone of diagram (22) is in \mathbf{MND}^{wk} , that is, all the truncation maps are weak. This can be seen in the same way as the analogous result for the strict case (see proof of Theorem 4.15)—the map $!$ is certainly weak, and D_M is seen to preserve weakness just as for F_M .
3. Moreover the underlying cone in \mathbf{CAT} of diagram (22) is

$$\begin{array}{ccccc} & \infty\text{-}\mathbf{Gph} & & & \\ & \swarrow \quad \searrow & & \searrow & \\ \dots & \longrightarrow & 2\text{-}\mathbf{Gph} & \xrightarrow{U_2} & 1\text{-}\mathbf{Gph} & \xrightarrow{U_1} & 0\text{-}\mathbf{Gph} \end{array}$$

so we know it is a limit in \mathbf{CAT} and at least a cone in \mathbf{CAT}_d .

4. So we can apply Proposition A.5 to conclude that the vertex $(\infty\text{-}\mathbf{Gph}, T_\infty)$ is in \mathbf{MND}_d , and that the cone

$$\begin{array}{ccccc} & (\infty\text{-}\mathbf{Gph}, T_\infty) & & & \\ & \swarrow \quad \searrow & & \searrow & \\ \dots & \longrightarrow & (2\text{-}\mathbf{Gph}, T_2^i) & \longrightarrow & (1\text{-}\mathbf{Gph}, T_1^i) & \longrightarrow & (0\text{-}\mathbf{Gph}, T_0^i) \end{array}$$

(tr₂, τ₂) (tr₁, τ₁) (tr₀, τ₀)

is a limit in both \mathbf{MND}_d and \mathbf{MND} . So we have shown (B).

We now prove (A). We already know that the underlying cone is in \mathbf{MND}_d and that the sequential diagram is in $(\mathbf{Top}, 1)/\mathbf{MND}_d$, so we just need to show that the vertex $(\infty\text{-}\mathbf{Gph}, T_\infty, \Pi_\infty)$ is in $(\mathbf{Top}, 1)/\mathbf{MND}_d$, and the only missing fact is that the functor

$$\mathbf{Top} \xrightarrow{\Pi_\infty} \infty\text{-}\mathbf{Cat}$$

is in \mathbf{CAT}_d not just \mathbf{CAT} . We do this by showing it is induced by the universal property of $\infty\text{-}\mathbf{Cat}$ in \mathbf{CAT}_d .

Recall that \mathbf{Alg} and \mathbf{Alg}_d each have a right adjoint and so preserve limits (Propositions 4.9 and 4.10); thus the diagram of categories of algebras

$$\begin{array}{c}
 \infty\text{-}\mathbf{Cat} \\
 \swarrow \quad \searrow \quad \searrow \quad \searrow \\
 \cdots \longrightarrow 2\text{-iCat} \longrightarrow 1\text{-iCat} \longrightarrow 0\text{-iCat}
 \end{array} \tag{23}$$

is a limit in both \mathbf{CAT}_d and \mathbf{CAT} (as usual we need the latter for the next proof); compare with Section 5. Furthermore we have another cone in \mathbf{CAT}_d coming from the underlying sequential diagram, that is we have

$$\begin{array}{c}
 \mathbf{Top} \\
 \swarrow \quad \searrow \quad \searrow \quad \searrow \\
 \cdots \longrightarrow 2\text{-iCat} \longrightarrow 1\text{-iCat} \longrightarrow 0\text{-iCat}
 \end{array}$$

Now the functor Π_∞ makes everything commute (see diagram (14), Definition 7.11), so must be the factorisation induced by the universal property of the above limit in \mathbf{CAT}_d . Thus we see that Π_∞ is also in \mathbf{CAT}_d as required.

This completes the proof that (22) is a limit; it remains to prove that this limit is preserved by

$$D_M: (\mathbf{Top}, 1)/\mathbf{MND}_d \longrightarrow (\mathbf{Top}, 1)/\mathbf{MND}_d.$$

Since the forgetful functor

$$(\mathbf{Top}, 1)/\mathbf{MND}_d \longrightarrow \mathbf{MND}_d$$

reflects limits, it is again enough to show that the underlying cone in \mathbf{MND}_d is a limit. We use Proposition A.5 again. We know:

1. The diagram is in \mathbf{MND}^{wk} as D_M preserves weakness.
2. The underlying cone is in \mathbf{CAT}_d as it underlies a cone in \mathbf{MND}_d .
3. The underlying cone is a limit in \mathbf{CAT}

$$\begin{array}{c}
 \infty\text{-Gph} \cong (\infty\text{-Gph})\text{-Gph} \\
 \swarrow \quad \searrow \quad \searrow \quad \searrow \\
 \cdots \longrightarrow 2\text{-Gph} \xrightarrow{U_2} 1\text{-Gph} \xrightarrow{U_1} 0\text{-Gph}
 \end{array}$$

so the cone in question is a limit in \mathbf{MND}_d (and \mathbf{MND}) as required. ■

We now deduce our other main theorem: that the category $\infty\text{-Cat}$ of Trimble ∞ -categories, together with the fundamental ∞ -groupoid functor Π_∞ , is the terminal coalgebra for D_C .

Proof of Theorem 7.14. We use Adámek's Theorem. We have already seen (Lemma 7.13) that it suffices to show that $(\mathbf{Top} \xrightarrow{\Pi_\infty} \infty\text{-Cat})$ is the limit in $\mathbf{Top}/\mathbf{CAT}_p$ of the diagram

$$\cdots \xrightarrow{U_3} \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_2^i \\ 2\text{-iCat} \end{array} \right) \xrightarrow{U_2} \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_1^i \\ 1\text{-iCat} \end{array} \right) \xrightarrow{U_1} \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_0^i \\ 0\text{-iCat} \end{array} \right) \quad (24)$$

and that D_C preserves this limit.

As the functor

$$\mathbf{Top}/\mathbf{CAT}_p \longrightarrow \mathbf{CAT}_p$$

reflects limits, it suffices to show that the cone

$$\begin{array}{c} \infty\text{-Cat} \\ \swarrow \quad \downarrow \quad \searrow \quad \searrow \\ \cdots \longrightarrow n\text{-iCat} \longrightarrow (n-1)\text{-iCat} \longrightarrow \cdots \longrightarrow 0\text{-iCat} \end{array} \quad (25)$$

is a limit in \mathbf{CAT}_p . Further, as

$$\mathbf{CAT}_p \hookrightarrow \mathbf{CAT}$$

reflects limits, it suffices to show that this is a limit in \mathbf{CAT} , which is true as \mathbf{Alg} preserves limits (in fact this limit was exhibited in the previous proof).

To show that $D_C: \mathbf{Top}/\mathbf{CAT}_p \longrightarrow \mathbf{Top}/\mathbf{CAT}_p$ preserves this limit, we use the commutative diagram

$$\begin{array}{ccccc} (\mathbf{Top}, 1)/\mathbf{MND}_d & \xrightarrow{D_M} & (\mathbf{Top}, 1)/\mathbf{MND}_d & \xrightarrow{U} & \mathbf{MND} \\ \downarrow \mathbf{Alg} & & \downarrow \mathbf{Alg} & & \downarrow \mathbf{Alg} \\ \mathbf{Top}/\mathbf{CAT}_d & \xrightarrow{D_C} & \mathbf{Top}/\mathbf{CAT}_d & \xrightarrow{U} & \mathbf{CAT} \end{array}$$

(Lemma 8.9). The image of the limit cone (25) under D_C is a cone in $\mathbf{Top}/\mathbf{CAT}_d$, which we wish to show is a limit cone in $\mathbf{Top}/\mathbf{CAT}_p$. By the usual Propositions 6.4 and A.3, it is enough to show that *its* underlying cone in \mathbf{CAT} is a limit. This cone is

$$UD_C \left(\left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_\infty \\ \infty\text{-Cat} \end{array} \right) \xrightarrow{\text{tr}_n} \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_n^i \\ n\text{-iCat} \end{array} \right) \right)_{n \geq 0}$$

$$\cong UD_C \mathbf{Alg} \left(\left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_\infty \\ \infty\text{-}\mathbf{Gph}^{T_\infty} \end{array} \right) \xrightarrow{(\mathrm{tr}_n, \tau_n)} \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_n^i \\ n\text{-}\mathbf{Gph}^{T_n^i} \end{array} \right) \right)_{n \geq 0}$$

But $UD_C \mathbf{Alg} \cong \mathbf{Alg} UD_M$, and in the proof of Theorem 8.13, we showed that

$$UD_M \left(\left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_\infty \\ \infty\text{-}\mathbf{Cat} \end{array} \right) \xrightarrow{\mathrm{tr}_n} \left(\begin{array}{c} \mathbf{Top} \\ \downarrow \Pi_n^i \\ n\text{-}\mathbf{iCat} \end{array} \right) \right)_{n \geq 0}$$

is a limit cone in \mathbf{MND} . Since $\mathbf{Alg}: \mathbf{MND} \longrightarrow \mathbf{CAT}$ preserves limits (Proposition 4.9), the result follows. \square

Finally we characterise T_∞ , the monad for Trimble weak ∞ -categories, directly. First we define a series of monads P_k on $\infty\text{-}\mathbf{Gph}$ inductively as follows; the idea is that P_k is the monad for “composition along bounding k -cells”.

- $P_0 = \mathbf{fc}_{(\infty\text{-}\mathbf{Gph}, \Pi_\infty E)}$
- for $k \geq 1$, $P_k = (P_{k-1})_* = (\mathbf{fc}_{(\infty\text{-}\mathbf{Gph}, \Pi_\infty E)})_{k*}$

where the subscript $k*$ is shorthand for $*$ applied k times.

Now, we cannot define T_∞ globally using these monads, as it would need to be an “infinite composite”. However, at each dimension n , the action of T_∞ on n -cells is given by a finite composite of the P_k .

8.14. THEOREM. T_∞ is given as follows. For an ∞ -graph A we have

- $(PA)_0 = A_0$, and
- for $n \geq 1$, $(T_\infty A)_n = (P_0 P_1 \cdots P_{n-1} A)_n$.

PROOF. By the limit cone (22) we know

$$(T_\infty A)_n = (T_n^i \mathrm{tr}_n A)_n$$

and by the definition of T_n^i (Definition 8.10) we know

- $T_0^i = id$
- $T_n^i = (T_{n-1}^i)^+ = \mathbf{fc}_{((n-1)\text{-}\mathbf{Gph}, \Pi_{n-1}^i E)} \circ (T_{n-1}^i)_*$.

Unravelling this we get

$$T_n^i = P_0^{(n)} P_1^{(n)} \cdots P_{n-1}^{(n)}$$

where $P_k^{(n)} = (\mathbf{fc}_{((n-k+1)\text{-}\mathbf{Gph}, \Pi_{n-k+1}^i E)})_{k*}$.

Thus we need to show that

$$(P_0 P_1 \cdots P_{n-1} A)_n = (P_0^{(n)} P_1^{(n)} \cdots P_{n-1}^{(n)} \mathrm{tr}_n(A))_n$$

Thus it suffices to show that for an n -graph B and any $k < n$

$$(P_k B)_n = (P_k^{(n)} B)_n$$

that is, on n -cells

$$(\mathbf{fc}_{(\infty\text{-}\mathbf{Gph}, \Pi_\infty E)})_{k*} = (\mathbf{fc}_{((n-k+1)\text{-}\mathbf{Gph}, \Pi_{n-k+1}^i E)})_{k*}$$

i.e. on $(n-k)$ -cells

$$\mathbf{fc}_{(\infty\text{-}\mathbf{Gph}, \Pi_\infty E)} = (\mathbf{fc}_{((n-k+1)\text{-}\mathbf{Gph}, \Pi_{n-k+1}^i E)}).$$

This follows from Remark 6.7, which tells us that

$$\mathrm{tr}_r(\Pi_\infty E) = \Pi_r^i E.$$

■

Finally note that in the language of [7] we now have an iterative theory of (incoherent) n -categories parametrised by the series of operads

$$\Pi_0^i E, \Pi_1^i E, \Pi_2^i E, \dots$$

This emphasises the way in which the single operad E is used to parametrise every type of composition in Trimble's theory of higher-dimensional categories.

A. Appendix: Limits of monads

In this appendix we gather together some technical results about monads. Our main purpose is to prove that the limits we need to calculate in \mathbf{MND}_d behave in the way one might expect. We begin by proving Proposition 4.10, that the chain of adjunctions in Proposition 4.9 restricts to a chain of adjunctions

$$\begin{array}{ccc} & \mathbf{MND}_d & \\ \uparrow & \uparrow & \uparrow \\ \mathbf{Und} & \begin{array}{c} \dashv \quad \mathbf{Inc} \quad \dashv \end{array} & \mathbf{Alg} \\ \downarrow & \downarrow & \downarrow \\ & \mathbf{CAT}_d & \end{array}$$

Proof of Proposition 4.10. We have to show that all three functors restrict to the subcategories concerned, and that given an object in one of the subcategories, the associated unit and counit morphisms also lie in the subcategory. The only nontrivial cases are that **Alg** restricts to a functor $\mathbf{MND}_d \rightarrow \mathbf{CAT}_d$ and that, for $(\mathcal{V}, T) \in \mathbf{MND}_d$, the counit morphism

$$\mathbf{IncAlg}(\mathcal{V}, T) = (\mathcal{V}^T, \mathrm{id}) \longrightarrow (\mathcal{V}, T)$$

lies in \mathbf{MND}_d .

To prove these statements, we use the fact that for any $(\mathcal{V}, T) \in \mathbf{MND}$, the forgetful functor $G^T: \mathcal{V}^T \rightarrow \mathcal{V}$ creates limits, and creates all colimits that T preserves. For $(\mathcal{V}, T) \in \mathbf{MND}_d$, then, G^T creates finite products and small coproducts. It follows that \mathcal{V}^T has all finite products and small coproducts, and since G^T reflects isomorphisms, \mathcal{V}^T is infinitely distributive. It also follows that G^T preserves and reflects finite products and small coproducts, from which it can be shown that the image under **Alg** of a morphism in \mathbf{MND}_d is in \mathbf{CAT}_d . Finally, the underlying functor of the counit morphism above is G^T , which preserves finite products and small coproducts. \square

We now build up some technical lemmas.

A.1. LEMMA. *Let \mathbb{I} be a category and $R: \mathbb{I} \rightarrow \mathbf{CAT}$ a functor. Then any limit cone on the diagram R is also a 2-limit cone.*

(We use “2-limit” in its traditional, strict, sense.)

PROOF. See Kelly [11]. An elementary direct proof is also straightforward. \blacksquare

Given a class \mathcal{K} of colimits and a class \mathcal{L} of limits, denote by $\mathbf{CAT}_{\mathcal{K}, \mathcal{L}}$ the subcategory of \mathbf{CAT} consisting of the categories with, and functors preserving, \mathcal{K} -colimits and \mathcal{L} -limits. The following lemma is easily verified and well known.

A.2. LEMMA. *The inclusion $\mathbf{CAT}_{\mathcal{K}, \mathcal{L}} \hookrightarrow \mathbf{CAT}$ reflects limits.* \square

The lemma can also be explained in terms of 2-monad theory. For a 2-monad S on a 2-category \mathcal{C} , let $\mathbf{Alg}^{\mathrm{wk}}(S)$ be the category of S -algebras and their weak (pseudo) morphisms. It can be shown that the forgetful functor $\mathbf{Alg}^{\mathrm{wk}}(S) \rightarrow \mathcal{C}$ reflects 2-limits. (Since the present explanation is not essential for our purpose, we omit the proof.) For a suitably-chosen 2-monad S on \mathbf{CAT} we have an equivalence $\mathbf{Alg}^{\mathrm{wk}}(S) \simeq \mathbf{CAT}_{\mathcal{K}, \mathcal{L}}$, under which the forgetful functor corresponds to the inclusion into \mathbf{CAT} . The lemma follows.

Let \mathbf{CAT}_c be the subcategory of \mathbf{CAT} consisting of the categories with, and functors preserving, small coproducts.

A.3. PROPOSITION. *The inclusions*

$$\mathbf{CAT}_p \hookrightarrow \mathbf{CAT}, \quad \mathbf{CAT}_c \hookrightarrow \mathbf{CAT}, \quad \mathbf{CAT}_d \hookrightarrow \mathbf{CAT}$$

all reflect limits.

PROOF. Lemma A.2 proves the first two cases. It also implies that the inclusion $\mathbf{CAT}_{\text{pc}} \hookrightarrow \mathbf{CAT}$ reflects limits, where \mathbf{CAT}_{pc} is the subcategory of \mathbf{CAT} consisting of the categories with, and functors preserving, finite products and small coproducts. The inclusion $\mathbf{CAT}_{\text{d}} \hookrightarrow \mathbf{CAT}_{\text{pc}}$ also reflects limits (being full and faithful), proving the third case. ■

For a 2-category \mathcal{C} , let $\mathbf{Mnd}(\mathcal{C})$ be the category of monads in \mathcal{C} and their lax morphisms, and let $\mathbf{Mnd}^{\text{wk}}(\mathcal{C})$ be the subcategory consisting of only the weak morphisms.

A.4. PROPOSITION. *Let \mathcal{C} be a 2-category, \mathbb{I} a category, and $(A_i, T_i)_{i \in \mathbb{I}}$ a diagram in $\mathbf{Mnd}^{\text{wk}}(\mathcal{C})$. Let*

$$\left((A, T) \xrightarrow{(P_i, \pi_i)} (A_i, T_i) \right)_{i \in \mathbb{I}} \quad (26)$$

be a cone in $\mathbf{Mnd}^{\text{wk}}(\mathcal{C})$. Suppose that

$$\left(A \xrightarrow{P_i} A_i \right)_{i \in \mathbb{I}} \quad (27)$$

is a 2-limit cone in \mathcal{C} . Then (26) is a limit cone in $\mathbf{Mnd}(\mathcal{C})$.

This result, whose proof follows shortly, has a partial explanation in terms of 2-monad theory. For many 2-categories \mathcal{C} , including \mathbf{CAT} , there is a 2-monad S on \mathcal{C} such that (with the obvious notation)

$$\mathbf{Alg}^{\text{wk}}(S) \cong \mathbf{Mnd}^{\text{wk}}(\mathcal{C}), \quad \mathbf{Alg}^{\text{lax}}(S) \cong \mathbf{Mnd}(\mathcal{C})$$

and S has rank. The forgetful functor $\mathbf{Alg}^{\text{wk}}(S) \rightarrow \mathcal{C}$ reflects 2-limits, so the proposition will follow as long as the inclusion $\mathbf{Alg}^{\text{wk}}(S) \hookrightarrow \mathbf{Alg}^{\text{lax}}(S)$ preserves limits. We know from [19] that this inclusion has a left biadjoint, and so preserves *bilimits*, and that the inclusion $\mathbf{Alg}^{\text{str}}(S) \hookrightarrow \mathbf{Alg}^{\text{lax}}(S)$ of *strict* algebras has a left adjoint, and so preserves limits. These facts are not quite what we need, but at least make the result unsurprising. In any case, the following proof is elementary.

PROOF. Let

$$\left((B, S) \xrightarrow{(Q_i, \kappa_i)} (A_i, T_i) \right)_{i \in \mathbb{I}}$$

be a cone in $\mathbf{Mnd}(\mathcal{C})$. Since (27) is a 2-limit, and in particular a limit, there is a unique morphism $\bar{Q}: B \rightarrow A$ such that

$$Q_i = \left(B \xrightarrow{\bar{Q}} A \xrightarrow{P_i} A_i \right)$$

for all $i \in \mathbb{I}$.

We claim that there exists a unique 2-cell $\bar{\kappa}$ such that

$$\begin{array}{ccc} B & \xrightarrow{\bar{Q}} & A \xrightarrow{P_i} A_i \\ s \downarrow & \nearrow \bar{\kappa} & \downarrow T \nearrow \pi_i \downarrow T_i \\ B & \xrightarrow{\bar{Q}} & A \xrightarrow{P_i} A_i \end{array} = \begin{array}{ccc} B & \xrightarrow{Q_i} & A_i \\ s \downarrow & \nearrow \kappa_i & \downarrow T_i \\ B & \xrightarrow{Q_i} & A_i \end{array}$$

for all $i \in \mathbb{I}$. Indeed, this equation is equivalent to

$$\begin{array}{ccc} B & \xrightarrow{\bar{Q}} & A \\ s \downarrow & \Downarrow \bar{\kappa} & \downarrow T \\ B & \xrightarrow{\bar{Q}} & A \xrightarrow{P_i} A_i \end{array} = \begin{array}{ccccc} B & \xrightarrow{\bar{Q}} & A & \xrightarrow{T} & A \\ s \downarrow & \searrow Q_i & \downarrow P_i & \Downarrow \pi_i^{-1} & \downarrow P_i \\ B & \searrow \bar{Q} & A_i & \xrightarrow{T_i} & A_i \\ & \searrow Q_i & \downarrow P_i & & \\ & A & \xrightarrow{P_i} & & A_i \end{array}$$

for all $i \in \mathbb{I}$ (using the fact that (P_i, π_i) is a *weak* morphism of monads). By the 2-limit property of (27), the claim holds as long as the 2-cells

$$\left(\begin{array}{ccc} & P_i T \bar{Q} & \\ B & \Downarrow & A_i \\ & P_i \bar{Q} S & \end{array} \right)_{i \in \mathbb{I}}$$

on the right-hand side form a morphism of cones. This is a straightforward check, proving the claim.

It is also straightforward to check that the pair $(\bar{Q}, \bar{\kappa})$ satisfies the axioms for a morphism of monads. The result follows. \blacksquare

We now deduce a result that we have used repeatedly for calculating limits in \mathbf{MND}_d and related slice categories. To a first approximation it says that the forgetful functor $\mathbf{MND} \rightarrow \mathbf{CAT}$ reflects limits. One might expect this by some kind of 2-monadicity, an object of \mathbf{MND} being an object of \mathbf{CAT} equipped with extra structure. But since the morphisms in \mathbf{MND} are *lax*, and since we are interested in \mathbf{MND}_d as well as \mathbf{MND} , the statement is more complicated.

A.5. PROPOSITION. *Let \mathbb{I} be a category, let $(\mathcal{A}_i, T_i)_{i \in \mathbb{I}}$ be a diagram in $\mathbf{MND}_d^{\text{wk}}$, and let*

$$\left((\mathcal{A}, T) \xrightarrow{(P_i, \pi_i)} (\mathcal{A}_i, T_i) \right)_{i \in \mathbb{I}} \quad (28)$$

be a cone in \mathbf{MND}^{wk} . Suppose that the cone

$$\left(\mathcal{A} \xrightarrow{P_i} \mathcal{A}_i \right)_{i \in \mathbb{I}} \quad (29)$$

lies in \mathbf{CAT}_d and is a limit cone in \mathbf{CAT} . Then:

i) $(\mathcal{A}, T) \in \mathbf{MND}_d$

ii) the cone (28) is a limit cone in both \mathbf{MND} and \mathbf{MND}_d .

PROOF. For (1), we have to show that $T: \mathcal{A} \rightarrow \mathcal{A}$ preserves small coproducts. Composing T with the cone (29) gives a cone

$$\left(\mathcal{A} \xrightarrow{P_i T} \mathcal{A}_i \right)_{i \in \mathbb{I}} \quad (30)$$

in **CAT**. In fact this cone lies in **CAT**_c, since for each $i \in \mathbb{I}$ we have an isomorphism

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{P_i} & \mathcal{A}_i \\ T \downarrow & \cong \swarrow \pi_i & \downarrow T_i \\ \mathcal{A} & \xrightarrow{P_i} & \mathcal{A}_i \end{array}$$

and the functors T_i and P_i are, by hypothesis, both coproduct-preserving.

The cone (29) is a limit in **CAT**_c as well as **CAT**, by Proposition A.3. Hence T , as the unique morphism of cones from (30) to (29) in **CAT**, preserves coproducts.

For (2), we know from Lemma A.1 and Proposition A.4 that the cone (28) is a limit in **MND**. We have to show that it is a limit in **MND**_d. In other words, we have to show that given an object $(\mathcal{B}, S) \in \mathbf{MND}_d$ and a morphism

$$(\overline{Q}, \overline{\kappa}): (\mathcal{B}, S) \rightarrow (\mathcal{A}, T)$$

in **MND** such that each composite

$$(\mathcal{B}, S) \xrightarrow{(\overline{Q}, \overline{\kappa})} (\mathcal{A}, T) \xrightarrow{(P_i, \pi_i)} (\mathcal{A}_i, T_i)$$

lies in **MND**_d, the original morphism $(\overline{Q}, \overline{\kappa})$ also lies in **MND**_d. But a morphism in **MND** lies in **MND**_d if and only if its underlying functor lies in **CAT**_d, so this follows from the fact that the inclusion **CAT**_d \hookrightarrow **CAT** reflects limits (Proposition A.3). ■

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